A Fundamental Formula. Suppose

(i) $\mathbf{X}$ is a continuous random n-vector;

(ii) g is a continuously differentiable real valued function on the range of X;

(iii) $Z = g(\mathbf{X})$;

(iv) $P(\mathbf{X} \in B) = 0$ where

$$B = \{\mathbf{x} \in \mathbf{rng}\,\mathbf{X} : \nabla \mathbf{g}(\mathbf{x}) = \mathbf{0}\}.$$

Then Z is continuous and

$$f_Z(z) = \int_{\{\mathbf{x}:g(\mathbf{x})=z\}} f_{\mathbf{X}}(\mathbf{x})/|\nabla g(\mathbf{x})|\ d\mathbf{x}.$$

We may extend this formula as follows. Suppose

$$A \subset \mathbf{R}^n, \quad g : A \longrightarrow \mathbf{R}, \quad \psi : A \longrightarrow \mathbf{R}, \quad z \in \mathbf{R}$$

are such that g is continuously differentiable, ψ is continuous and

$$\nabla g(\mathbf{x}) \neq \mathbf{0} \quad \text{whenever} \quad \mathbf{x} \in A, g(\mathbf{x}) = z \text{ and } \psi(\mathbf{x}) \neq 0.$$

Suppose

$$B \subset \mathbf{R}^{n-1}, \quad f : B \longrightarrow \mathbf{R} \quad \text{is continuously differentiable,} \quad \{\mathbf{x} : g(\mathbf{x}) = z\} = f.$$

Then

$$\int_{\{\mathbf{x}:g(\mathbf{x})=z\}} \psi(\mathbf{x})/|\nabla g(\mathbf{x})|\ d\mathbf{x} = \int_A \psi(\mathbf{u}, f(\mathbf{u}))/|\frac{\partial g}{\partial x_n}(\mathbf{u}, f(\mathbf{u}))|\ d\mathbf{u}.$$