The probability generating function. Compare with p. 151ff in the book.

Suppose X is a random variable with *postive* values. For each real number t we let

$$\Phi_X(t) = E(t^X) \quad \text{whenever } t \in \mathbf{R};$$

note that because t^X is nonnegative and that, for some t, we could have $E(t^X) = \infty$. In particular, the values of X are nonnegative integers we have

$$\Phi_X(t) = \sum_{n=0}^{\infty} P(X = n)t^n.$$

The function Φ_X is called the **probability generating function of X.** Note that, at least formally,

$$\Phi_X^{(m)} = E\big(X(X-1)\cdots(X-(m-1))\big) \quad \text{whenever } m \text{ is a nonnegative integer.}$$

In particular,

$$\Phi_X(1) = 1; \quad \Phi'_X(1) = E(X), \quad \Phi''_X = E(X(X-1)) = E(X^2) - E(X);$$

it follows that

$$Var(X) = E(X(X-1)) + E(X) - E(X)^2 = \Phi''_X + \Phi_X(1) - \Phi_X(1)^2.$$

Example. Suppose X is binomial with parameters n and p. Then

$$\Phi_X(t) = \sum_{m=0}^{n} \binom{n}{m} p^m q^{n-m} t^m = (pt+q)^n.$$

Thus

$$E(X) = \Phi'_X(1) = np \quad \text{and} \quad E(X(X-1)) = \Phi''_X(1) = n(n-1)p^2$$

so

$$Var(X) = n(n-1)p^2 + np - (np)^2 = npq.$$

Example. Suppose X is geometric with parameter p. Then

$$\Phi_X(t) = \sum_{n=1}^{\infty} q^{n-1}pt^n = pt\sum_{n=1}^{\infty}(tq)^{n-1} = \frac{pt}{1-qt}.$$

Then

$$\Phi'_X(t) = \frac{p}{(tq-1)^2}, \quad \Phi''_X(t) = \frac{2pq}{(tq-1)^3}.$$

Thus

$$E(X) = \frac{1}{p}, \quad E(X(X-1)) = \frac{2q}{p^2}$$

so

$$Var(X) = \frac{2q}{p^2} + \frac{1}{p} - \big(\frac{1}{p}\big)^2 = \frac{q}{p^2}.$$

Example. Suppose X is Poisson with parameter λ. then

$$\Phi_X(t) = \sum_{n=0}^{\infty} e^{-\lambda}\frac{\lambda^n}{n!}t^n = e^{-\lambda}e^{\lambda t \lambda} = e^{\lambda(t-1)}.$$

Then

$$\Phi'_X(t) = \lambda e^{\lambda(t-1)}, \quad \Phi''_X(t) = \lambda^2$$

so

$$E(X) = \lambda, \quad E(X(X-1)) = \lambda^2$$

and

$$Var(X) = \lambda^2 + \lambda - \lambda^2 = \lambda.$$