

Linear transformations and Gaussian random vectors.

Remember, n -vectors are the same as $n \times 1$ matrices.

Let \mathbf{X} a random n -vector. We let

$$E(\mathbf{X})$$

be the n -vector whose i -th entry is $E(X_i)$. If \mathbf{Y} is a random n -vector we let

$$\text{Cov}(\mathbf{X}, \mathbf{Y})$$

be the $n \times n$ matrix whose i, j entry is $\text{Cov}(X_i, Y_j)$ and we let

$$\text{Var}(\mathbf{X}) = \text{Cov}(\mathbf{X}, \mathbf{X}).$$

We have already proved the simple

Proposition. Suppose \mathbf{X} is a random n -vector, A is an $n \times n$ matrix, \mathbf{b} is an n -vector and

$$\mathbf{Y} = A\mathbf{X} + \mathbf{b}.$$

Then

$$E(\mathbf{Y}) = AE(\mathbf{X}) + \mathbf{b}$$

and

$$\text{Var}(\mathbf{Y}) = A\text{Cov}(\mathbf{X})A^T.$$

Definition. We say the random vector \mathbf{X} is **standard normal** if its components X_1, \dots, X_n are independent and standard normal. Evidently, this is the case if and only if \mathbf{X} is continuous and

$$f_{\mathbf{X}}(\mathbf{x}) = (2\pi)^{-n/2} e^{-|\mathbf{x}|^2/2} \quad \text{for any } n\text{-vector } \mathbf{x}.$$

We say the random vector \mathbf{Y} is **Gaussian** if

$$\mathbf{Y} = A\mathbf{X} + \mathbf{b}$$

for some standard normal \mathbf{X} and nonsingular A . Note that

$$E(\mathbf{Y}) = \mathbf{b} \quad \text{and} \quad \text{Var}(\mathbf{Y}) = AA^T.$$

Discussion. Let \mathbf{X} be standard normal and suppose

$$\mathbf{Y} = A\mathbf{X} + \mathbf{b}$$

for some nonsingular A , so that \mathbf{Y} is Gaussian. Let $B = \sqrt{AA^T} = \sqrt{\text{Var}(\mathbf{Y})}$; B is well defined because AA^T is symmetric positive definite. If $\mathbf{y} = A\mathbf{x} + \mathbf{b}$ we have

$$f_{\mathbf{Y}}(\mathbf{y}) = (2\pi)^{-n/2} e^{-|\mathbf{x}|^2/2} |\det A|^{-1} = (2\pi)^{-n/2} |\det A|^{-1} e^{-|A^{-1}(\mathbf{y}-\mathbf{b})|^2/2}.$$

But

$$\begin{aligned} |A^{-1}(\mathbf{y} - \mathbf{b})|^2 &= (A^{-1}(\mathbf{y} - \mathbf{b}))^T A^{-1}(\mathbf{y} - \mathbf{b}) \\ &= (\mathbf{y} - \mathbf{b})^T (AA^T)^{-1} (\mathbf{y} - \mathbf{b}) \\ &= (B^{-1}(\mathbf{y} - \mathbf{b}))^T B^{-1}(\mathbf{y} - \mathbf{b}) \\ &= |B^{-1}(\mathbf{y} - \mathbf{b})|^2 \end{aligned}$$

and

$$\det A = \sqrt{\det AA^T} = \det B.$$

Thus we have the following

Theorem. Suppose \mathbf{Y} is a random n -vector. Then \mathbf{Y} is Gaussian if and only if $\text{Var}(\mathbf{Y})$ is positive definite and

$$f_{\mathbf{Y}}(\mathbf{y}) = (2\pi)^{-n/2} (\det \sqrt{\text{Var}(\mathbf{Y})})^{-1} e^{-|(\sqrt{\text{Var}(\mathbf{Y})})^{-1}(\mathbf{y} - E(\mathbf{Y}))|^2/2}$$

for any n -vector \mathbf{y} in which case

$$\mathbf{X} = (\sqrt{\text{Var}(\mathbf{Y})})^{-1}(\mathbf{y} - E(\mathbf{Y}))$$

is standard normal.