The Gamma and Poisson distributions and the Poisson process.

1. The basic setup.

We fix a positive number λ and a i.i.d. sequence

$$I_1, I_2, \ldots, I_n, \ldots$$

of random variables such that

$$P(I_1 > 0) = 1. \tag{1}$$

We let $T_0 = 0$ and we let

$$T_n = \sum_{i=1}^{n} I_n \quad \text{for each } n \in \mathbb{N}^+.$$

So if we think of I_n, $n = 1, 2, 3, \ldots$ as a *sequence of interarrival times* then T_n, $n \in \mathbb{N}^+$, is the corresponding *sequence of arrival times.* Owing to (1) we find that

$$0 < T_1 < T_2 < \cdots < T_n < \cdots \quad \text{with probability 1.}$$

We define random variables

$$N_t, \quad t \geq 0$$

with values in $\mathbb{N} \cup \{\infty\}$ by letting $N_0 = 0$ and requiring that, when $t > 0$, we have $P(N_t \notin \mathbb{N}) = 0$ and

$$\{N_t = n\} = \{T_n < t \leq T_{n+1}\} \quad \text{whenever } n \in \mathbb{N};$$

in other words, N_t is the number of arrivals before time t; this is possible because of (1). Equivalently, when $t > 0$,

$$\{N_t \geq n\} = \{T_n < t\} \quad \text{for any } n \in \mathbb{N}.$$

The following Lemma will make everything work. The way you get anywhere in this business is to condition on T_1; this amounts to starting over at $t = T_1$. Whatever that means.

Lemma 1.1. Suppose

(i) $N \in \mathbb{N}^+$;
(ii) $t_0 = 0$, $(t_1, \ldots, t_N) \in (0, \infty)^N$ and

$$t_1 < \cdots < t_N;$$

(iii) $(n_1, \ldots, n_N) \in \mathbb{N}^n$;
(iv) $m_i = \sum_{j=1}^{i} n_j$ for $i \in \{1, \ldots, N\}$;
(v) $I \in \{0, 1, \ldots, N\}$ is such that

$$m_i \begin{cases} = 0 & \text{if } i \leq I, \\ > 0 & \text{if } i > I \end{cases} \qquad \text{whenever } i \in \{1, \ldots, N\}.$$

Then

$$\cap_{i=1}^{N} \{N_{t_i} = n_i\} = \cap_{i=1}^{N} \{N_{t_i} - N_{t_{i-1}} = m_i\} \tag{2}$$

and, for any $u \in (0,\infty)$,

$$
\begin{aligned}
&P\big(\cap_{i=1}^{N}\{N_{t_i}=m_i\}|T_1=u\big)\\
&\qquad=\begin{cases} P(\cap_{i=I+1}^{N}\{N_{t_i-u}=m_i-1\}\big) & \text{if } I<N \text{ and } t_I<u<t_{I+1},\\ 1 & \text{if } I=N \text{ and } u<t_1,\\ 0 & \text{if } u\ge t_N.\end{cases}
\end{aligned}
\tag{3}
$$

Remark 1.1. Given $m_i \in \mathbb{N}$, $i \in \{0,1,\ldots,N\}$ with $0=m_0\le m_1\le\cdots\le m_N$ we can set

$$n_i=m_i-m_{i-1}\in\mathbb{N} \quad \text{for } i\in\{1,\ldots,N\}.$$

Proof. (2) is evident. As to (3), we let

$$A=\{i\in\{1,\ldots,N\}: i\le I\} \quad \text{and we let} \quad B=\{i\in\{1,\ldots,N\}: i>I\};$$

we observe that for any $i\in\{1,\ldots,N\}$ and $0<u<\infty$ we have

$$
\begin{aligned}
&\{N_{t_i}=m_i\}\cap\{T_1=u\}\\
&\quad=\{T_{m_i}<t_i\le T_{m_i+1}\}\cap\{T=u\}\\
&\quad=\begin{cases}\{0<t_i<u\}\cap\{T=u\} & \text{if } i\in A,\\ \{T_{m_i}-T_1<t_1-u\le T_{m_i+1}-T_1\}\cap\{T_1=u\} & \text{if } i\in B;\end{cases}\\
&\quad=\begin{cases}\emptyset & \text{if } i\in A \text{ and } u\le t_i,\\ \{T=u\} & \text{if } i\in A \text{ and } u>t_i,\\ \emptyset & \text{if } i\in B \text{ and } u\ge t_i,\\ \{T_{m_i}-T_1<t_i-u\le T_{m_i+1}-T_1\}\cap\{T_1=u\} & \text{if } i\in B \text{ and } u<t_i.\end{cases}
\end{aligned}
$$

It follows that if $I<N$ and $0<u<\infty$ then

$$
\begin{aligned}
&P\big(\cap_{i=1}^{N}\{N_{t_i}=m_i\}|T_1=u\big)\\
&\qquad=\begin{cases}P(\cap_{i=I+1}^{N}\{T_{m_i-1}<t_i-u\le T_{m_i}\}\big) & \text{if } u<t_1,\\ 0 & \text{if } u\ge t_1\end{cases}
\end{aligned}
$$

and that if $I=N$ and $0<u<\infty$ then

$$
\begin{aligned}
&P\big(\cap_{i=1}^{N}\{N_{t_i}=m_i\}|T_1=u\big)\\
&\qquad=\begin{cases}1 & \text{if } u\ge t_1,\\ 0 & \text{if } u<t_1.\end{cases}
\end{aligned}
$$

□

2. The definitions of the relevant distributions.

Definition 2.1. Suppose $\lambda\in(0,\infty)$. We set

$$p_\lambda(n)=e^{-\lambda}\frac{\lambda^n}{n!} \quad \text{for } n\in\mathbb{N}.$$

We say the random variable N has the **Poisson distribution with parameter** λ if

$$P(N=n)=p_\lambda(n) \quad \text{whenever } n\in\mathbb{N};$$

note that

$$P(N\notin\mathbb{N})=0 \quad \text{since} \quad \sum_{n=0}^{\infty}\frac{\lambda^n}{n!}=e^\lambda.$$

Suppose $n \in \mathbb{N}^+$. We say the random variable G has the **gamma distribution with parameters n and λ** if G is continuous and $f_G = \Gamma_n$ where we have set

$$\Gamma_n(x) = \lambda \begin{cases} p_{\lambda x}(n-1) & \text{if } x > 0, \\ 0 & \text{if } x \le 0. \end{cases}$$

Obviously, $\Gamma_n \ge 0$; we will show presently that the integral of Γ_n is 1.

Theorem 2.1. Suppose $n \in \mathbb{N}^+$; $0 < \lambda < \infty$; $X_1, \ldots, X_n$ are independent random variables each of which has the exponential distribution with parameter λ and $G = \sum_{i=1}^n X_i$. Then G has the gamma distribution with parameters n and λ.

Proof. We induct on n. It's immediate for $n = 1$ since Γ_1 is the pdf of a random variable with parameter λ. Suppose $n \in \mathbb{N}^+$; the Theorem for n; $X_1, \ldots, X_n, X_{n+1}$ are independent random variables each of which has the exponential distribution with parameter λ; and $G = \sum_{i=1}^n X_i$. Since X_{n+1}, G are independent we have for any $x > 0$

$$\begin{aligned} f_{G+X_{n+1}}(x) &= f_{X_{n+1}} * f_G(x) \\ &= \int_0^x \lambda e^{-\lambda(x-y)} \left(\lambda \frac{(\lambda y^{n-1}}{(n-1)!}\right) dy \\ &= \lambda^{n+1} e^{-\lambda x} \int_0^x \frac{y^{n-1}}{(n-1)!}\, dy \\ &= \lambda \frac{(\lambda x)^{(n+1)-1}}{((n+1)-1)!} \\ &= \Gamma_{n+1}(x). \end{aligned}$$

Since $P(G + X_{n+1} < 0) = 0$ we are done. $\square$

3. The basic theorem.

The remainder of this article is devoted to the proof of the following Theorem.

Theorem 3.1. The following conditions are equivalent:

(i) For any $N \in \mathbb{N}^+$ and

$$t_1 < t_2 < \cdots < t_N$$

the random variables

$$N_{t_1}, N_{t_2} - N_{t_1}, \ldots, N_{t_N} - N_{t_{N-1}}$$

are independent.

(ii) For some $\lambda > 0$, I_1 is exponentially distributed with parameter λ.

Moreover, when these conditions hold, $N_t - N_s$ is Poisson with parameter $\lambda(t-s)$ whenever $0 < s < t$.

Proof. Suppose $N_t - N_s, N_s$ are independent whenever $0 < s < t$. Let $f(s) = P(N_s = 0)$ for $s > 0$. Then

$$f(s+t) = P(N_t - N_s = 0, N_s = 0) = P(N_t = 0)P(N_s = 0) = f(s)f(t)$$

whenever $s, t > 0$. Since $f(s) = P(I_1 > s)$ we find that f is continuous and decreasing and that $\lim_{s \downarrow 0} f(s) = 1$. It follows that $f(s) = e^{-\lambda s}$ for $s \in (0, \infty)$ so that I_1 has the exponential distribution with parameter λ.

Let

$$f(t,n) = e^{-\lambda t}\frac{(\lambda t)^n}{n!} \quad \text{for } t > 0 \text{ and } n \in \mathbb{N}.$$

Thus the Theorem will be proved once we show that

$$P\big(\cap_{i=1}^N \{N_{t_i} = m_i\big) = \prod_{i=1}^N f(t_i - t_{i-1}, m_i - m_{i-1}) \tag{4}$$

whenever $t_0 = 0$, $(t_1, \ldots, t_N) \in (0,\infty)^N$, $m_0 = 0$, $(m_1, \ldots, m_N) \in \mathbb{N}^n$,

$$t_1 < \cdots < t_N \quad \text{and} \quad m_1 \leq \cdots \leq m_N.$$

Indeed, if $n_1, \ldots, n_N \in \mathbb{N}$ and $m_i = \sum_{j=1}^i n_j$ then

$$\begin{aligned}\{N_{t_1} = n_1, N_{t_2} - N_{t_1} = n_2, \ldots, N_{t_N} - N_{t_{N-1}} = n_N\}\\ = \{N_{t_1} = m_1, N_{t_2} = m_2, \ldots, N_{t_N} = m_N\}.\end{aligned}$$

We prove (4) by conditioning on T_1.

We being by noting that (4) holds whenever $m_i = 0$ for $i \in \{1, \ldots, N\}$.

We begin by establishing (4) in case $N = 1$. (4) clearly holds when $N = 1$ and $m_1 = 0$. Suppose $m_1 > 0$ and (4) holds with $N = 1$ and with m_1 replaced by any smaller nonnegative integer. We have

$$\begin{aligned} P(N_{t_1} = m_1) &= \int_0^\infty P(N_{t_1} = m_1 | T_1 = u) f(u,0)\,du \\ &= \int_0^{t_1} P(N_{t_1 - u} = m_1 - 1) f(u,0)\,du \\ &= \int_0^{t_1} f(t_1 - u, m_1 - 1) f(u,0)\,du \\ &= \int_0^{t_1} e^{-\lambda(t_1-u)}\frac{(\lambda(t_1-u))^{n-1}}{(n-1)!} e^{-\lambda u}\,du \\ &= f(t_1, m_1) \end{aligned}$$

Thus (4) holds when $N = 1$.

Suppose $N \in \mathbb{N}^+$ and (4) holds when N is replaced by any smaller positive integer. We induct on m_1.

In case $m_1 = 0$ we find that (4) holds by making use of Lemma 1.1. Letting $I = \max\{i \in \{1, \ldots, N\} : m_i = 0\} > 0$ we find that, in case $I < N$,

$$\begin{aligned}
&P(N_{t_1} = 0, N_{t_2} = m_2, \dots, N_{t_N} = m_N) \\
&\qquad = \int_0^\infty P(N_{t_1} = 0, N_{t_2} = m_2, \dots, N_{t_N} = m_N | T_1 = u) f(u,0)\, du \\
&\qquad = \int_{t_I}^{t_{I+1}} P(\cap_{i=I+1}^{N} \{N_{t_i - u} = m_i - 1\}) f(u,0)\, du \\
&\qquad = \prod_{i=I+1<i\leq N} f(t_i - t_{i-1}, m_i - m_{i-1}) \int_{t_I}^{t_{I+1}} f(t_{I+1} - u, m_{I+1} - 1) f(u,0)\, du \\
&\qquad = \prod_{I+1<I\leq N} f(t_i - t_{i-1}, m_i - m_{i-1}) f(t_{I+1}, m_{I+1}) \\
&\qquad = \prod_{i=1}^{N} f(t_i - t_{i-1}, m_i - m_{i-1})
\end{aligned}$$

and that, in case $I = N$, $m_i = 0$ for $i \in \{1, \dots, N\}$ and

$$\begin{aligned}
&P(N_{t_1} = 0, N_{t_2} = 0, \dots, N_{t_N} = 0) \\
&\qquad = P(N_{t_N} = 0) \\
&\qquad = f(t_N - t_0, 0) \\
&\qquad = \prod_{i=1}^{N} f(t_i - t_{i-1}, m_i - m_{i-1}).
\end{aligned}$$

So suppose $m_1 > 0$ and (4) holds when m_1 is replaced by any nonnegative integer less than m_1. Using Lemma 1.1 with I there equal 0 we find that

$$\begin{aligned}
&P(N_{t_1} = m_1, N_{t_2} = m_2, \dots, N_{t_N} = m_N) \\
&\qquad = \int_0^\infty P(N_{t_1} = m_1, N_{t_2} = m_2, \dots, N_{t_N} = m_N | T_1 = u) f(u,0)\, du \\
&\qquad = \int_0^{t_1} P(N_{t_1 - u} = m_1 - 1, N_{t_2 - u} = m_2 - 1, \dots, N_{t_N - u} = m_N - 1) f(u,0)\, du \\
&\qquad = \prod_{i=2}^{N} f(t_i - t_{i-1}, m_i - m_{i-1}) \int_0^{t_1} f(t_1 - u, m_1 - 1) f(u,0)\, du \\
&\qquad = \prod_{i=1}^{N} f(t_i - t_{i-1}, m_i - m_{i-1}).
\end{aligned}$$

□