Problem 25 on page 55.

A pair of fair dice are rolled until a sum of five or seven appears. Find the probability that a five appears first.

Solution One.

Let's first consider one roll of the dice. Let us use $T = \{1,2,3,4,5,6\}^2$ for the sample space. Note that $|T| = 6^2 = 36$. Let

$$Q(E) = \frac{|E|}{|T|} = \frac{|E|}{36} \quad \text{whenever } E \subset T.$$

Let Five=$\{(1,4),(2,3),(3,2),(4,1)\}$, let Seven=$\{(1,6,(2,5),(3,4),(4,3),(5,2),(1,6)\}$ and let Neither=$T \sim$ (Five $\cup$ Seven). Thus

$$Q(\text{Five}) = \frac{4}{36} = \frac{1}{9}, \quad Q(\text{Seven}) = \frac{6}{36} = \frac{1}{6}, \quad Q(\text{Neither}) = \frac{36-(4+6)}{36} = \frac{13}{18}.$$

Now consider rolling the dice until a sum of five or seven appears. For each positive integer n let

$$S_n = \text{Neither}^{n-1} \times (\text{Five} \cup \text{Seven})$$

and let

$$S = \cup_{n=1}^{\infty} S_n.$$

Then S is a reasonable sample space for this experiment. Let P be a "reasonable" probability for this experiment. It turns out to be somewhat technical to spell out exactly what this is; I will make below what I hope are immediately intuitive assertion about P of certain events. For each positive integer n let

$$E_n = \text{Neither}^{n-1} \times \{\text{Five}\}.$$

Then

$$P(E_n) = Q(\text{Neither})^{n-1} Q(\text{Five}).$$

Here is a good way to think about this. A fundamental property of this experiment is that the result of one roll of the dice is *independent* of the result of another throw of the dice. Thus

$$\begin{aligned} P(E_n) &= P(\text{Neither}_1 \cap \cdots \cap \text{Neither}_{n-1} \cap \text{Five}_n) \\ &= P(\text{Neither}_1) \cdots P(\text{Neither}_{n-1}) P(\text{Five}_n) \\ &= P(\text{Neither}_1)^{n-1} P(\text{Five}_n); \end{aligned}$$

here the subscript i means "on the i-th roll".

Let

$$E = \cup_{n=1}^{\infty} E_n.$$

This is the event that that a five appears first. We have

$$P(E) = \sum_{n=1}^{\infty} P(E_n) = \sum_{n=1}^{\infty} (\frac{13}{18})^{n-1} \frac{1}{9} = \frac{1}{9} \frac{1}{1 - \frac{13}{18}} = \frac{2}{5}.$$

Solution Two using conditioning. We have

$$P(E) = P(E|\mathrm{Five}_1)P(\mathrm{Five}_1) + P(E|\mathrm{Neither}_1)P(\mathrm{Neither}_1)\,.$$

Evidently,

$$P(\mathrm{Five}_1) = Q(\mathrm{Five}) = \frac{1}{9}, \quad P(\mathrm{Neither}_1) = Q(\mathrm{Neither}) = \frac{13}{18}.$$

It should also be clear that

$$P(E|\mathrm{Five}_1) = 1.$$

The main point is that

$$P(E|\mathrm{Neither}_1) = P(E);$$

I hope this makes sense to you. Thus

$$P(E) = Q(\mathrm{Five}) + P(E)Q(\mathrm{Neither})$$

so

$$P(E) = \frac{Q(\mathrm{Five})}{1 - Q(\mathrm{Neither})} = \frac{2}{5}.$$