**Mathematics 135.01. Final Exam. Fall 1997.**

**TO GET FULL CREDIT YOU MUST SHOW ALL WORK!**

**I have neither given nor received aid in the completion of this test.**
**Signature:**

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**1. 10 pts.** Suppose the events $A, B, C, D, E, F$ are mutually independent with probabilities $a, b, c, d, e, f$, respectively. Calculate

$$P((A \cup B) \cap (C \cup D) \cap (E \cup F))$$

in terms of $a, b, c, d, e, f$.

**Solution.** Since the events $A \cap B,\ C \cap D,\ E \cap F$ are independent we have

$$P(A \cup B) \cap (C \cup D) \cap (E \cup F)) \;=\; P(A \cup B)P(C \cup D)P(E \cup F).$$

Since the events $A$ and $B$ are independent we have

$$P(A \cup B) \;=\; P(A) + P(B) - P(A \cap B) \;=\; P(A) + P(B) - P(A)P(B) \;=\; a + b - ab.$$

One treats $P(C \cup D)$ and $P(E \cup F)$ similarly. Thus

$$P((A \cup B) \cap (C \cup D) \cap (E \cup F)) \;=\; (a + b - ab)(c + d - cd)(e + f - ef).$$

**2. 10 pts.** Suppose $X$ is uniform on $(0, 1)$, $Y$ is exponential with parameter 1 and $X$ and $Y$ are independent. Compute the pdf of $X/Y$.

**Solution.** Since $X$ and $Y$ are independent we have that $(X, Y)$ is continuous and $f_{X,Y}(x, y) = f_X(x)f_Y(y)$. Thus

$$f_{X,Y}(x, y) \;=\; \begin{cases} e^{-y} & \text{if } 0 < x < 1 \text{ and } 0 < y, \\ 0 & \text{else.} \end{cases}$$

Let $Z = X/Y$ and note that the range of $Z$ is $(0, \infty)$. For $z \in (0, \infty)$ we let $R_z = \{(x, y) : x/z \le y\}$ and calculate

$$\begin{aligned} P(Z \le z) &= P(X/Y \le z) \\ &= \int\int_{R_z} f_{X,Y}(x, y)\, dxdy \\ &= \int_0^1 \Big(\int_{x/z}^{\infty} e^{-y} dy\Big) dx \\ &= \int_0^1 e^{-x/z}\, dx \\ &= z(1 - e^{-\frac{1}{z}}). \end{aligned}$$

Differentiating this we find that

$$f_Z(z) \;=\; \begin{cases} 1 - e^{-\frac{1}{z}}(1 + \frac{1}{z}) & \text{if } z > 0 \\ 0 & \text{else.} \end{cases}$$

**3. 10 pts.** Suppose $(X, Y)$ is uniformly distributed over $\{(x, y) : 0 < x < y < 1\}$. Compute $\mathrm{Cov}(X, Y)$.

**Solution.** Note that the area of $\{(x, y) : 0 < x < y < 1\}$ is 2 so

$$f_{X,Y}(x, y) \;=\; \begin{cases} 2 & \text{if } 0 < x < y < 1, \\ 0 & \text{else.} \end{cases}$$

We have

$$\begin{aligned} \mathrm{E}(X) \;&=\; \int_0^1 \Big( \int_x^1 x\,2\,dx \Big) dy \;=\; \frac{1}{3}; \\ \mathrm{E}(Y) \;&=\; \int_0^1 \Big( \int_x^1 y, 2\,dx \Big) dy \;=\; \frac{2}{3}; \\ \mathrm{E}(XY \;&=\; \int_0^1 \Big( \int_x^1 xy, 2\,dx \Big) dy \;=\; \frac{1}{4}. \end{aligned}$$

Thus $\mathrm{Var}(XY) = \frac{1}{4} - \frac{1}{3}\frac{2}{3} = \frac{1}{36}$.

**4. 10 pts.** Suppose $Y$ is what comes up when a fair die is rolled and suppose $X$ is chosen uniformly from the nonnegative integers not exceeding twice $Y$. Compute the pmf of $(X, Y)$.

Suggestion: Compute $p_{X|Y}$ first.

**Solution.** For each $y = 1, 2, 3, 4, 5, 6$ we have

$$p_{X|Y}(x|y) \;=\; \begin{cases} \frac{1}{2y+1} & \text{if } x = 0, 1, 2, \ldots, 2y, \\ 0 & \text{else.} \end{cases}$$

For these same $y$ we have $p_{X,Y}(x, y) = p_{X|Y}(x|y) p_Y(y)$ for any $x$ so

$$p_{X,Y}(x, y) \;=\; \begin{cases} \frac{1}{6(2y+1)} & \text{if } y = 1, 2, 3, 4, 5, 6 \text{ and } x = 0, 1, 2, \ldots, 2y, \\ 0 & \text{else.} \end{cases}$$

**5. 10 pts.** Suppose $X$ is uniformly distributed on $(0, 1)$. Calculate the probability density function of $1/X$.

**Solution.** Let $Y = 1/X$ and note that the range of $Y$ is $(1, \infty)$. Given $y \in (1, \infty)$ we have

$$F_Y(y) \;=\; P(Y \le y) \;=\; P(1/X \le y) \;=\; P(1/y \le X) \;=\; 1 - 1/y.$$

Differentiating, we conclude that

$$f_Y(y) \;=\; \begin{cases} \frac{1}{y^2} & \text{if } 1 < y, \\ 0 & \text{else.} \end{cases}$$

**6. 15 pts.** A certain device consists of $n$ components of a certain type which operate independently of each other and each of whose time to failure is exponentially distributed with parameter $\lambda > 0$. The device functions as long as least two of the components are functioning. Compute the pdf of the time to failure of the device.

Hint: Let $T$ be the time to failure of the device. Let $E_{i,j}$, $1 \le i < j \le n$, be the event that components $i$ and $j$ are the last two components to fail. Given $t > 0$, what is $P(T > t|E_{i,j})$?

**Solution.** The Hint is dumb. It could lead you an incorrect solution of the Problem. However, I did warn you that I was suspicious of the Hint during the test.

For each $i = 1, \ldots, n$ let $X_i$ be the time to failure of the $i$-the component. Suppose $t > 0$. Then $\{T \le t\}$ is the disjoint union of the event that all of the $X_i$ have failed by time $t$ whose probability is $(1 - e^{-\lambda t})^n$ with the event that exactly one of the $X_i$ is still functioning at time $t$ whose probability is $n\, e^{-\lambda t}(1 - e^{-\lambda t})^{n-1}$. Thus

$$F_T(t) = P(T \le t) = (1 - e^{-\lambda t})^n + n\, e^{-\lambda t}(1 - e^{-\lambda t})^{n-1}.$$

Differentiating, we find that

$$f_T(t) = \begin{cases} \lambda n(n-1)e^{-2\lambda t}(1 - e^{-\lambda t})^{n-1} & \text{if } t > 0, \\ 0 & \text{else.} \end{cases}$$

**7. 10 pts.** Suppose $(X, Y, Z)$ is uniformly distributed on $(0,1) \times (0,1) \times (0,1)$. Compute $P(X < Y < Z^2)$.

**Solution.** The answer is the volume of $\{(x, y, z) : 0 < x < y < z^2 < 1\}$ which is

$$\int_0^1 \text{Area}\{(x, y) : 0 < x < y < z^2\}\, dz = \int_0^1 \frac{z^4}{2}\, dz = \frac{1}{10}.$$

**8. 15 pts.** Suppose $X$ is uniform on $(0,1)$ and $Y$ is uniform on $(0, X^2)$. Compute the joint pdf of $(X, Y)$.

Hint: Compute $f_{Y|X}$ first.

**Solution.** For $x \in (0,1)$ we have

$$f_{Y|X}(y|x) = \begin{cases} \frac{1}{x^2} & \text{if } 0 < y < x^2, \\ 0 & \text{else.} \end{cases}$$

For these same $x$ we have $f_{X,Y} = f_{Y|X}(y|x) f_X(x)$ for any $y$. Thus

$$f_{X,Y}(x, y) = \begin{cases} \frac{1}{x^2} & \text{if } 0 < x < 1 \text{and } 0 < y < x^2, \\ 0 & \text{else.} \end{cases}$$

**9. 20 pts.** Suppose $N$ is the number of flips of a fair coin until the first head. Suppose $Y$ is uniform on $(0, N)$. Suppose $X$ is exponential with parameter $\sqrt{Y}$. Compute the expectation of $X$.

Suggestion: Use the conditional expectation formula.

Remark: The answer is an infinite series which is not to hard to sum; you don't have to sum it to get full credit.

**Solution.** We first note that We have

$$\mathrm{E}(X) \;=\; \mathrm{E}(\mathrm{E}(X|Y)) \;=\; \mathrm{E}(\frac{1}{\sqrt{Y}}) \;=\; \mathrm{E}(\mathrm{E}(\frac{1}{\sqrt{Y}}|N)).$$

Now for any positive integer $n$ we have

$$\mathrm{E}(\frac{1}{\sqrt{Y}}|N=n) \;=\; \frac{1}{n}\int_0^n \frac{1}{\sqrt{y}}\,dy \;=\; \frac{2}{\sqrt{n}}.$$

Thus $\mathrm{E}(\frac{1}{\sqrt{Y}}|N) = \frac{2}{\sqrt{N}}$ so

$$\mathrm{E}(X) \;=\; \mathrm{E}(\frac{2}{\sqrt{N}}) \;=\; 2\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}2^n}.$$

I don't know how to sum this. It turns that I had intended to have $Y$ uniform on $(0, N^2)$ and not $(0, N)$, but I made a typo. If you take what I intended then, by the same method, you get

$$\mathrm{E}(X) \;=\; 2\sum_{n=1}^{\infty}\frac{1}{n2^n} \;=\; 2\ln 2.$$

**10. 20 pts.** Suppose $S$ is number of heads in 100 flips of a fair coin. Suppose $T$ is the number of heads in 200 flips of a coin where the probability of getting heads on a given flip is $\frac{1}{4}$. Use the Central Limit Theorem to compute approximately the probability that $S < T + 10$.

Hint: For each $i = 1, 2, \ldots, 100$ let $X_i$ be the indicator random variable for the event the $i$-th flip of the first coin is a head, and for each $i = 1, 2, \ldots, 200$ let $Y_i$ be the indicator random variable for the event the $i$-th flip of the second coin is a head. Note that $S - T = \sum_{i=1}^{100} S_i - T_{2i-1} - T_{2i}$. If you think this is kloogy, and you very well might, you could use the version of the Central Limit Theorem on page 406.

**Solution.** For each $i = 1, 2, \ldots$ let $Z_i = X_i - Y_{2i-1} - Y_{2i}$. We have that

$$\mathrm{E}(Z_i) \;=\; \mathrm{E}(X_i) - \mathrm{E}(Y_{2i-1}) - \mathrm{E}(Y_{2i}) \;=\; \frac{1}{2} - \frac{1}{4} - \frac{1}{4} \;=\; 0$$

and, by our independence assumptions, that

$$\mathrm{Var}(Z_i) \;=\; \mathrm{Var}(X_i) + \mathrm{Var}(Y_{2i-1}) + \mathrm{Var}(Y_{2i}) \;=\; \frac{1}{2}\frac{1}{2} + \frac{1}{4}\frac{3}{4} + \frac{1}{4}\frac{3}{4} \;=\; \frac{5}{8}.$$

Note that the $Z_i$'s are independent. Using the Central Limit Theorem we find that

$$\begin{aligned}
P(S < T + 10) \;&=\; P(\sum_{i=1}^{100} Z_i < 10.5) \\
&=\; P(\frac{\sum_{i=1}^{100} Z_i - 100\cdot 0}{\sqrt{100\cdot\frac{5}{8}}} < \frac{10.5 - 100\cdot 0}{\sqrt{100\cdot\frac{5}{8}}}) \\
&\approx\; \Phi(\frac{10.5 - 100\cdot 0}{\sqrt{100\cdot\frac{5}{8}}}) \\
&\approx\; \Phi(1.33) \\
&\approx\; .9082.
\end{aligned}$$