.

1. The Fourier transform.

For $f : \mathbb{R} \to \mathbb{C}$ with $\int |f(x)|\,dx < \infty$ we let

$$\hat{f} : \mathbb{R} \to \mathbb{C}$$

be such that

$$\hat{f}(\xi) = \int e^{-ix\xi} f(x)\,dx \quad \text{for } \xi \in \mathbb{R}.$$

Why do we care about this? Because if X is a continuous random variable with

$$\int |x| f_X(x)\,dx$$

then

$$\xi_X(t) = E\left(e^{itX}\right) = \int e^{itx} f_X(x)\,dx = \widehat{f_X}(-t) \quad \text{for } t \in \mathbb{R}.$$

And you can do essentially the same thing when X is discrete. We want to show, among other things, that if two random variables have the same characterisic function then they have the same distribution. I guess that's why they call ξ_X the characteristic function of X.

For $f : \mathbb{R} \to \mathbb{C}$ we define

$$\tau_a f : \mathbb{R} \to \mathbb{C} \quad \text{for } a \in \mathbb{R}$$

by

$$\tau_a f(x) = f(x - a) \quad \text{for } x \in \mathbb{R}.$$

One verifies that

$$\widehat{\tau_a f}(\xi) = e^{-ia\xi} \hat{f}(\xi) \quad \text{for } \xi \in \mathbb{R}.$$

This implies that

$$\hat{f}'(\xi) = -i\xi \hat{f}(\xi) \quad \text{for } \xi \in \mathbb{R}.$$

For $f : \mathbb{R} \to \mathbb{C}$ we define

$$\delta_r f : \mathbb{R} \to \mathbb{C} \quad \text{for } r \in (0, \infty)$$

by

$$\delta_r f(x) = f\left(\frac{x}{r}\right) \quad \text{for } x \in \mathbb{R}.$$

One verifies that

$$\widehat{\delta_r f}(\xi) = r\hat{f}(r\xi) \quad \text{for } \xi \in \mathbb{R}.$$

We have

$$\int f(x)\hat{g}(x)\,dx = \int \hat{f}(\xi) g(\xi)\,d\xi.$$

This implies

$$\begin{aligned}
\int f(\epsilon x)\hat{g}(x)\,dx &= \int \delta_{1/\epsilon} f(x)\hat{g}(x)\,dx \\
&= \int \widehat{\delta_{1/\epsilon} f}(\xi) g(\xi)\,d\xi \\
&= \int \frac{1}{\epsilon}\hat{f}\left(\frac{\xi}{\epsilon}\right) g(\xi)\,d\xi \\
&= \int \hat{f}(\eta) g(\epsilon\eta)\,d\eta
\end{aligned}$$

where in the last step we made the substitution $\xi = \epsilon\eta$, Letting $\epsilon \downarrow 0$ we obtain

$$f(0)\int \hat{g}(x)\,dx = g(0)\int \hat{f}(\eta)\,d\eta.$$

Replacing f by $\tau_{-x}f$ we find that

$$f(x)\int \hat{g}(x)\,ds = g(0)\int e^{ix\xi}\hat{f}(\xi)\,d\xi$$

for $x \in \mathbb{R}$; so we are oh so close to inverting the Fourier transform!

The Gaussian. Let

$$g(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2} \quad \text{for } x \in \mathbb{R}.$$

We have

$$\begin{aligned} \sqrt{2\pi}\left(\frac{d}{d\xi}+\xi\right)&\hat{g}(\xi) \\ &= \left(\frac{d}{d\xi}+\xi\right)\int e^{-ix\xi - x^2/2}\,dx \\ &= \int (-ix+\xi)e^{-ix\xi-x^2/2}\,dx \\ &= i\int e^{-ix\xi-x^2/2}\,d_x(-ix\xi - x^2/2) \\ &= 0. \end{aligned}$$

Thus

$$\hat{g}(\xi) = \hat{g}(0)e^{-\xi^2/2} \quad \text{for } \xi \in \mathbb{R}.$$

Now

$$\hat{g}(0) = \frac{1}{\sqrt{2\pi}}\int e^{-x^2/2}\,dx = 1$$

so

$$\hat{g}(\xi) = e^{-\xi^2/2} \quad \text{for } \xi \in \mathbb{R}.$$

Thus we have established the Fourier Inversion Formula:

$$f(x) = \frac{1}{2\pi}\int e^{-ix\xi}\hat{f}(x)\,dx \quad \text{for } x \in \mathbb{R}.$$

I'll leave it to you to check that all these integrals converge!

2. Back to characteristic functions and such.

Proposition 2.1. Suppose $a \in \mathbb{R} \sim \{0\}$, $b \in \mathbb{R}$, X is a random variable and $Y = aX + b$. Then

$$\chi_Y(t) = e^{ib}\chi_X(at) \quad \text{for } t \in \mathbb{R}.$$

Proof. Turn the crank. □

Definition 2.1. We say the random variable Z is **standard normal** if Z is continuous and

$$f_Z = g$$

where g is the Gaussian introduced above.

It follows from the foregoing that

$$\chi_Z(t) = E\left(e^{itZ}\right) = \int_{-\infty}^{\infty} e^{itx} f_Z(z)\, dz = \hat{g}(-t) = e^{-t^2/2} \quad \text{for } t \in \mathbb{R}.$$

Since

$$\frac{d}{dt}^m E(e^{itX})\Big|_{t=0} = i^m E(X^m) \quad \text{for any nonnegative integer } m$$

we find that

$$E(Z) = 0 \quad \text{and} \quad Var(Z) = 1.$$

Definition 2.2. Suppose $\mu, \sigma \in \mathbb{R}$ and $\sigma > 0$. We say the random variable is **normal with parameters** μ **and** σ^2 if X has the same distribution as $\sigma Z + \mu$ where Z is standard normal.

It follows that $E(X) = \mu$, $Var(X) = \sigma^2$, X is continuous and

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2} \quad \text{for } x \in \mathbb{R}.$$

(Why?)

Proposition 2.2. Suppose X and Y are independent random variables. Then

$$\chi_{X+Y} = \chi_X \chi_Y.$$

Proof. For any $t \in \mathbb{R}$ we have

$$E(e^{it(X+Y)}) = E(e^{itX} e^{itY}) = E(e^{itX}) E(e^{itY}).$$

□

Theorem 2.1. Suppose $X_1, \ldots, X_n$ are independent random variables and, for each $j = 1, \ldots, n$, X_j is normal with parameters μ_j and σ_j^2. Then $Y = \sum_{j=1}^n X_j$ is normal with parameters $\sum_{j=1}^n \mu_j$ and $\sum_{j=1}^n \sigma_j^2$.

Proof. For each $j = 1, \ldots, n$ let $Z_j = (X_j - \mu_j)/\sigma_j$ and note that Z_j is standard normal. Evidently $Y = sum_{j=1} \sigma_j Z_j + \mu_j$ and $\sigma_j Z_1 + \mu_1, \ldots, \sigma_n Z_n + \mu_n$ are independent. Thus

$$\begin{aligned}
\chi_Y(t) &= \chi_{\sigma_1 Z_1 + \mu_1}(t) \cdots \chi_{\sigma_n Z_n + \mu_n}(t) \\
&= e^{i\mu_1} \chi(\sigma_1 t) \cdots e^{i\mu_n} \chi(\sigma_n t) \\
&= e^{i\mu_1} e^{-(\sigma_1 t)^2/2} \cdots e^{i\mu_n} e^{-(\sigma_n t)^2/2} \\
&= e^{i \sum_{j=1}^n \mu_j} e^{-((\sum_{j=1}^n \sigma_j)^2)/2} \\
&= \chi_W(t)
\end{aligned}$$

if W is normal with parameters $\sum_{j=1}^n \mu_j$ and $\sum_{j=1}^n \sigma_j^2$. □