**Computing expectation by conditioning.**

Let $(S, \mathcal{E}, P)$ be a probability space, let $F \in \mathcal{E}$ be such that

$$P(F) > 0.$$

Suppose $X$ is a discrete random variable. We let

$$E(X|F)$$

be the expectation of $X$ with respect to the probability $P(\cdot|F)$. It follows that

$$E(X|F) = \sum_x x P(X = x|F). \tag{1}$$

Moreover, if $F_1, \ldots, F_n \in \mathcal{E}$ are such that $F_i \cap F_j = \emptyset$ whenever $i \neq j$ and $S = \cup_{i=1}^n F_i$ then, as one may easily verify,

$$E(X) = \sum_{i=1}^n E(X|F_i)P(F_i). \tag{2}$$

**Example.** Let $X_1, X_2, X_3, \ldots$ be a sequence of independent Bernoulli random variable with parameter $p > 0$. Let $q = 1 - 0$ and let

$$G = \min\{n : X_n \neq 0\}$$

so $G$ is geometric with parameter $p$.

**Proposition.** We have

$$P(G = n|X_1 = 0) = P(G + 1 = n) \quad \text{and} \quad P(G = n|X_1 = 1) = P(1 = n) \quad \text{for any positive integer } n. \tag{3}$$

**Proof.** We have

$$P(G = 1|X_1 = 1) = P(X_1 = 1|X_1 = 1) = 1 = P(1 = 1)$$

and

$$P(G = 1|X_1 = 0) = P(X_1 = 1|X_1 = 0) = 0 = P(G + 1 = 1)$$

so these equations hold if $n = 1$. If $n > 1$ we have

$$P(G = n|X_1 = 1) = P(X_1 = 0, \ldots, X_{n-1} = 0, X_n = 1|X_1 = 1) = 0 = P(1 = n)$$

and

$$\begin{aligned} P(G = n|X_1 = 0) &= P(X_1 = 0, \ldots, X_{n-1} = 0, X_n = 1|X_1 = 1) \\ &= \frac{P(X_1 = 0, \ldots, X_{n-1} = 0, X_n = 1, X_1 = 0)}{P(X_1 = 0)} \\ &= \frac{P(X_1 = 0) \cdots P(X_{n-1} = 0)P(X_n = 1)}{P(X_1 = 0)} \\ &= q^{n-2}p \\ &= P(G = n - 1) \\ &= P(G + 1 = n). \end{aligned}$$

$\square$

We infer that

$$E(G|X_1 = 0) = E(G + 1) = E(G) + 1 \quad \text{and} \quad E(G|X_1 = 1) = E(1) = 1.$$

It follows from (2)that

$$E(G) = E(G|X_1 = 0)P(X_1 = 0) + E(G|X_1 = 1)P(X_1 = 1) = (E(G) + 1)q + 1\,p$$

which gives

$$E(G) = \frac{1}{p}.$$

We obtain from (3) that, for any positive integer $n$,

$$P(G^2 = n^2|X_1 = 0) = P(G = n|X_1 = 0) = P(G + 1 = n|X_1 = 0) = P((G + 1)^2 = n^2|X_1 = 0)$$

and

$$P(G^2 = n^2|X_1 = 1) = P(G = n|X_1 = 1) = P(1 = n) = P(1 = n^2)$$

It follows from (2) that

$$E(G^2) = E(G^2|X_1 = 0)P(X_1 = 0) + E(G^{|}X_1 = 1)P(X_1 = 1) = (E(G^2) + 2E(G) + 1)q + p$$

which gives

$$E(G) = \frac{1}{p} \quad \text{and} \quad E(G^2) = \frac{2 - p}{p^2}.$$

In particular,

$$Var(G) = E(G^2) - E(G)^2 = \frac{q}{p^2}.$$