Problem. Suppose (X, Y) is uniformly distributed on the square $S = \{(x, y) : 0 < x < 1 \text{ } and \text{ } 0 < y < 1\}$. Ascertain wheter or not XY is continuous and if it is calculate its pdf.

Solution. Let $Z = XY$ and let $g(x, y) = xy$ for $(x, y) \in \mathbf{R}^2$. We have $\nabla g(x, y) = (y, x)$ for $(x, y) \in \mathbf{R}^2$ so

$$P((X, Y) \in \{(x, y) : \nabla g(x, y) = (0, 0)\} = P((X, Y) = (0, 0)) = 0.$$

Thus Z is continuous and

$$f_Z(z) = \int_{\{(x,y):xy=z\}} f_{(X,Y)}(x, y)/|\nabla g(x, y)| \; d(x, y).$$

Inasmuch as

$$f_{(X,Y)}(x, y) = \begin{cases} 1 & \text{if } (x, y) \in S, \\ 0 & \text{else} \end{cases}$$

we see that the above integral equals 0 when $z < 0$ or $1 < z$ and that it equals

$$\int_{\{(x,y)\in S:xy=z\}} 1/|\nabla g(x, y)| \; d(x, y)$$

otherwise. Now fix z with $0 < z < 1$ and observe that

$$\{(x, y) \in S : xy = z\} = \{(x, z/x) : z < x < 1\}.$$

Since $\frac{\partial g}{\partial y}(x, y) = x$ we have

$$f_Z(z) = \int_z^1 1/x \; dx = -\ln(z).$$

Putting this all together we have

$$f_Z(z) = \begin{cases} -ln(z) & \text{if } 0 < z < 1, \\ 0 & \text{else.} \end{cases}$$