The expectation of a continuous random variable.

Suppose X is a continuous random variable. For each positive integer N let

$$X_{N,-} = \sum_{n \in \mathbb{Z}} \frac{n}{N} 1_{\{n/N \le X < (n+1)/N\}}$$

and let

$$X_{N,+} = \sum_{n \in \mathbb{Z}} \frac{(n+1)}{N} 1_{\{n/N \le X < (n+1)/N\}}.$$

Evidently,

$$X_{N,-} \le X \le X_{N,+} \quad \text{and} \quad 0 \le X_{N,+} - X_{N,-} \le \frac{1}{N}. \tag{1}$$

Now

$$\begin{aligned}
E(X_{N,-}) &= \sum_{n \in \mathbb{Z}} \frac{n}{N} P(\{n/N \le X < (n+1)/N\}) \\
&= \sum_{n \in \mathbb{Z}} \frac{n}{N} \int_{n/N}^{(n+1)/N} f_X(x)\,dx \\
&\le \sum_{n \in \mathbb{Z}} \int_{n/N}^{(n+1)/N} x f_X(x)\,dx \\
&= \int_{-\infty}^{\infty} x f_X(x)\,dx \\
&= \sum_{n \in \mathbb{Z}} \int_{n/N}^{(n+1)/N} x f_X(x)\,dx \\
&\le \sum_{n \in \mathbb{Z}} \frac{(n+1)}{N} P(\{n/N \le X < (n+1)/N\}) \\
&= E(X_{N,+}).
\end{aligned}$$

Thus, keeping in mind (1) which implies that $E(X_{N,+} - X_{N,-}) \le 1/N$, we find that

$$\lim_{N \to \infty} E(X_{N,-}) = \int_{-\infty}^{\infty} x f_X(x)\,dx = \lim_{N \to \infty} E(X_{N,+})$$

leading us to *define*

$$E(X) = \int_{-\infty}^{\infty} x f_X(x)\,dx.$$