It's dark and you're trying to open the door...

We begin with a very useful formula.

The multiplication rule. (See page 68 in the book.) Suppose $E_1, \ldots, E_m$ are events. Then

$$P(E_m \cap \cdots \cap E_1) = P(E_m|E_{m-1} \cap \cdots \cap E_1)P(E_{m-1}|E_{m-2} \cap \cdots \cap E_1) \cdots P(E_2|E_1)P(E_1).$$

Proof. Just apply the definition of conditional probability and watch everything cancel. $\square$

Problem. Suppose it's dark and you have n keys to n different doors in your pocket and your trying to open one of these doors. You take keys out of your pocket and try them until, as will surely happen, the key opens the door. You're taking Mathematics 135 so you're pretty smart and you *don't* put a key that doesn't work back in the pocket that originally had all the keys in it. Suppose $k \in \{1, \ldots, n\}$. What is the probability that the k-th key pulled out works?

Solution. For each $i \in \{1, \ldots, n\}$ let E_i be the event that the i-th key pulled out works and let F_i be the complementary event. Applying the multiplication rule we find that

$$\begin{aligned} P(E_k) &= P(E_k \cap F_{k-1} \cap \cdots \cap F_1) \\ &= P(E_k|F_{k-1} \cap \cdots \cap F_1)P(F_{k-1}|F_{k-2} \cap \cdots \cap F_1) \cdots P(F_2|F_1)P(F_1) \\ &= \frac{1}{n-(k-1)} \frac{n-(k-2)-1}{n-(k-2)} \cdots \frac{(n-1)-1}{n-1} \frac{n-1}{n} \\ &= \frac{1}{n}. \end{aligned}$$

$\square$