Counting.

Whenever X is a set we let

$$|X|$$

be the number of members of X if X is finite and we let it be ∞ if X is infinite. Here are some basic counting principles.

1. Unions. Suppose $\mathcal{X}$ is a disjointed family of finite sets. Then

$$|\bigcup \mathcal{X}| = \sum_{X \in \mathcal{X}} |X|. \tag{1}$$

For example, if m is a positive integer and $X_1, \ldots, X_m$ are finite sets such that

$$X_i \cap X_j = \emptyset \qquad \text{whenever } 1 \le i < j \le m \tag{2}$$

we can let $\mathcal{X} = \{X_1, \ldots, X_m\}$ and obtain

$$|\bigcup_{i=1}^{m} X_i| = \sum_{i=1}^{m} |X_i|. \tag{3}$$

Even if (2) does not hold we have the **inclusion-exclusion priciple**

$$|\bigcup_{i=1}^{m} X_i| = \sum_{j=1}^{m} (-1)^{j-1} \sum_{1 \le i_1 < \cdots < i_j \le m} |X_{i_1} \cap \cdots \cap X_{i_j}|. \tag{4}$$

which can be proved by induction on m.

2. Summing over the range of a function. Here is an important special case of (1). If $f : X \to Y$, we may set

$$\mathcal{X} = \{\{x \in X : f(x) = y\} : y \in Y\} = \{f^{-1}[\{y\}] : y \in Y\}$$

in (1) to obtain

$$|X| = \sum_{y \in Y} |\{x \in X : f(x) = y\}|. \tag{5}$$

3. Products. Suppose m is a positive integer and $X_1, \ldots, X_m$ are sets. Then

$$X_1 \times \cdots \times X_m$$

is, by definition, the set of m-tuples

$$(x_1, \ldots, x_m)$$

such that $x_i \in X_i,\ i = 1, \ldots, m$. We have

$$|X_1 \times \cdots \times X_m| = |X_1| \cdots |X_m| \tag{6}$$

with the understanding that 0 times ∞ is 0.

4. Exponentiation. If X and Y are sets we let

$$Y^X$$

be the set of functions whose domain equals X and whose range is a subset of Y. If X and Y are finite then

$$|Y^X| = |Y|^{|X|}. \tag{7}$$

5. Families of subsets. Suppose X is a set. Whenever m is a nonnegative integer not exceeding $|X|$ we let

$$\binom{X}{m} = \{A : A \subset X \text{ and } |A| = m\}.$$

That is, $\binom{X}{m}$ is the family of m-member subsets of X. Whenever $m_1, \ldots, m_k$ are nonnegative integers such that

$$\sum_{i=1}^{k} m_i = |X|$$

we let

$$\binom{X}{m_1 \; \cdots \; m_k} = \{(A_1, \ldots, A_k) : A_i \subset X,\ |A_i| = m_i \ i = 1, \ldots, k \text{ and } A_i \cap A_j = \emptyset,\ 1 \le i < j \le k\}.$$

6. Nonrepeating sequences. Whenever X is a set and m is a positive integer not exceeding $|X|$ we let

$$(X)_m = \{(x_1, \ldots, x_m) \in \underbrace{X \times \cdots \times X}_{m \text{ times}} : x_i \ne x_j \text{ whenever } 1 \le i < j \le m\}.$$

7. Some basic definitions of counts. We let

$$\binom{n}{m}, \qquad \binom{n}{m_1 \; \cdots \; m_k}, \qquad (n)_m$$

equal

$$|\binom{X}{m}|, \qquad |\binom{X}{m_1 \; \cdots \; m_k}|, \qquad |(X)_m|,$$

respectively, where $X = \{1, \ldots, n\}$.

Theorem. We have

$$\binom{n}{m} = \frac{n!}{m!\,(n-m)!}, \qquad \binom{n}{m_1 \; \cdots \; m_k} = \frac{n!}{m_1! \cdots m_k!}, \qquad (n)_m = \frac{n!}{(n-m)!}. \tag{8}$$

Proof. Let $X = \{1, \ldots, n\}$. One proves the third formula by induction on n making use of (5) applied to the function $f : (X)_m \to X$ which sends $(x_1, \ldots, x_m)$ to x_m to show that $(n)_m = n\,(n-1)_{m-1}$. One proves the first formula by making use of (5) and the third formula applied to the function from $(X)_m$ to $\binom{X}{m}$ which sends $(x_1, \ldots, x_m)$ to $\{x_1, \ldots, x_m\}$. To prove the second formula we induct on k making use of (5) applied to the function

$$f : \binom{X}{m_1 \; \cdots \; m_k} \to \binom{X}{m_k}$$

which sends $(A_1, \cdots, A_k)$ to A_k to infer that

$$\binom{n}{m_1 \ \cdots \ m_k} = \binom{n - m_k}{m_1 \ \cdots \ m_{k-1}} \binom{n}{m_k}.$$

□

8. Example. Suppose $A = \{a, b, c, d\}$ where a, b, c, d are distinct objects. Then

$$\begin{aligned}
\binom{A}{n} &= \emptyset \quad \text{if } n \text{ is an integer greater than } 4; \\
\binom{A}{4} &= \{\{a, b, c, d\}\}; \\
\binom{A}{3} &= \{\{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}; \\
\binom{A}{2} &= \{\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}\}; \\
\binom{A}{1} &= \{\{a\}, \{b\}, \{c\}, \{d\}\}; \\
\binom{A}{0} &= \{\emptyset\}.
\end{aligned}$$

Note that the family

$$\left\{ \binom{A}{4}, \binom{A}{3}, \binom{A}{2}, \binom{A}{1}, \binom{A}{0} \right\}$$

is disjointed with union 2^A; in particular

$$2^4 = \binom{4}{4} + \binom{4}{3} + \binom{4}{2} + \binom{4}{1} + \binom{4}{0}$$

or

$$16 = 1 + 4 + 6 + 4 + 1.$$

Poker.

Here are some definitions.

$$\begin{aligned}
\text{Kinds} &= \{2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A\}; \\
\text{Suits} &= \{C, D, H, S\}; \\
\text{Deck} &= \text{Kinds} \times \text{Suits}; \\
\text{PokerHands} &= \binom{\text{Deck}}{5}.
\end{aligned}$$

Thus

$$|\text{PokerHands}| = \binom{|13||4|}{5} = \binom{52}{5} = 2,598,960.$$

There is a natural univalent map from OnePair onto

$$\{(k, S, T, U) : k \in \text{Kinds}, \ S \in \binom{\text{Suits}}{2}, \ T \in \binom{\text{Kinds} \sim \{k\}}{3} \text{ and } U \in \text{Suits}^3\}.$$

Thus

$$|\text{OnePair}| \;=\; \sum_{k\in\text{Kinds}} \binom{4}{2}\binom{12}{3}4^3 \;=\; 13\binom{4}{2}\binom{12}{3}4^3 \;=\; 1,098,240.$$

There is a natural univalent map from TwoPair onto

$$\{(R,S,c) : R \in \binom{\text{Kind}}{2},\ S \in \binom{\text{Suits}}{2}^2,\ c \in (\text{Kinds} \sim R) \times \text{Suits}\}.$$

Thus

$$|\text{TwoPair}| \;=\; \sum_{R\in\binom{\text{Kind}}{2}} \binom{4}{2}^2 11\cdot 4 \;=\; \binom{13}{2}\binom{4}{2}^2 11\cdot 4 \;=\; 123,552.$$

There is a natural univalent map from ThreeOfAKind onto

$$\{(k,T,U,s) : k \in \text{Kinds},\ T \in \binom{\text{Suits}}{3},\ U \in \binom{\text{Kinds} \sim \{k\}}{2} \text{ and } s \in \text{Suits}^2\}.$$

Thus

$$|\text{ThreeOfAKind}| \;=\; \sum_{k\in\text{Kinds}} \binom{4}{3}\binom{12}{2}4^2 \;=\; 13\binom{4}{3}\binom{12}{2}4^2 \;=\; 54,912.$$

There is a natural map from Straight∪StraighFlush onto

$$\{A,2,3,4,5,6,7,8,9,10,\} \times \text{Suits}^5.$$

Thus, as Straight is disjoint from StraightFlush,

$$|\text{Straight}| + |\text{StraightFlush}| \;=\; 10\cdot 4^5 \;=\; 10,240$$

so

$$|\text{Straight}| \;=\; 11,264 - |\text{StraightFlush}|$$

There is a natural univalent map from Flush∪StraightFlush onto $\binom{\text{Kinds}}{5} \times \text{Suits}$. Thus, as Flush is disjoint from StraighFlush,

$$|\text{Flush}| + |\text{StraightFlush}| \;=\; \binom{13}{5}4 \;=\; 5,148$$

so

$$|\text{Flush}| \;=\; 5,148 - |\text{StraightFlush}|.$$

There is a natural univalent map from FullHouse onto

$$\{(S,T,U) : S \in (\text{Kinds})_2,\ T \in \binom{\text{Suits}}{3} \text{ and } U \in \binom{\text{Suits}}{2}\}.$$

Thus

$$|\text{FullHouse}| \;=\; 13\,12\binom{4}{3}\binom{4}{2} \;=\; 3,744.$$

There is natural univalent map from FourOfAKind to

$$\{(k,l,s) : k \in \text{Kind},\ l \in \text{Kind} \sim \{k\} \text{ and } s \in \text{Suit}\}$$

so that

$$|\text{FourOfAKind}| \;=\; \sum_{k\in\text{Kind}} 12\cdot 4 \;=\; 13\cdot 12\cdot 4 \;=\; 624.$$

There is a natural univalent map from StraightFlush onto

$$\{A, 2, 3, 4, 5, 6, 7, 8, 9, 10\} \times \text{Suits}$$

so that

$$|\text{StraightFlush}| \;=\; 10 \cdot 4 \;=\; 40.$$

It is obvious that

$$|\text{RoyalFlush} \;=\; 4.$$

Thus

$$\begin{aligned}
P(\text{OnePair}) \;&=\; \frac{1,098,240}{2,598,960} \approx .4226 \\
P(\text{TwoPair}) \;&=\; \frac{123,552}{2,598,960} \approx .0475 \\
P(\text{ThreeOfAKind}) \;&=\; \frac{54,912}{2,598,960} \approx .02113 \\
P(\text{Straight}) \;&=\; \frac{10,024 - 40}{2,598,960} \approx .00392 \\
P(\text{Flush}) \;&=\; \frac{5,148 - 40}{2,598,960} \approx .001966 \\
P(\text{FullHouse}) \;&=\; \frac{3,744}{2,598,960} \approx .00144 \\
P(\text{FourOfAKind}) \;&=\; \frac{624}{2,598,960} \approx .0002401 \\
P(\text{StraightFlush}) \;&=\; \frac{40}{2,598,960} \approx .0000153 \\
P(\text{RoyalFlush}) \;&=\; \frac{4}{2,598,960} \approx .000001539.
\end{aligned}$$