Problem 3.1(18), again. Let

$$\Omega$$

be the set of

$$((r_1^1, \dots, r_1^n), (r_2^1, \dots, r_2^n), (r_3^1, \dots, r_3^n))$$

where, for each $i \in \{1, 2, 3\}$, $r_i^j \in \{1, \dots, n\}$ for $j \in \{1, \dots, n\}$ and $r_i^j \neq r_i^k$ for $j, k \in \{1, \dots, n\}$ with $j \neq k$. Note that

$$|\Omega| = (n!)^3.$$

We let

$$P(E) = \frac{|E|}{|\Omega|} \quad \text{for } E \subset \Omega. \tag{1}$$

Thus P is the **uniform probability distribution on Ω.** In other words, two events are equally likely if they have same number of members.

For each $i \in \{1, 2, 3\}$ and each $j \in \{1, \dots, n\}$ let

$$R_i^j$$

be the rank committee member i gives to applicant j. A basic fact about this setup is that whenever $j_1, j_2, j_3 \in \{1, \dots, n\}$ we have

$$P(R_1^{j_1} = r_1, R_2^{j_2}, P(R_3^{j_3} = r_3) = P(R_1^{j_1} = r_1)P(R_2^{j_2} = r_2)P(R_3^{j_3} = r_3)$$

whenever $i \in \{1, 2, 3\}$ and $r_1, r_2, r_3\{1, \dots, n\}$. That is, in terminology we shall introduce shortly, *the random variables $R_1^{j_1}, R_2^{j_2}, R_3^{j_3}$ are independent*; **independence** is the most important concept in probability. Indeed,

$$P(R_1^{j_1} = r_1, R_2^{j_2}, R_3^{j_3} = r_3) = \frac{((n-1)!)^3}{(n!)^3} = \frac{1}{n^3} \tag{2}$$

and

$$P(R_i^j = k) = \frac{(n-1)!(n!)^2}{(n!)^3} = \frac{1}{n}. \tag{3}$$

One of you, essentially, took (2) and (3) as a starting point for the problem and then argued as I do in what follows; his intuition was quite sound but I objected that he hadn't used an appropriate sample space. The real problem is to figure out exactly what "rank the candidates randomly" means. It means (1) above. As we move away from these *ad hoc* counting problems we will see the sample space recede into the background and will see it no more. The notion of random variable and its associated distribution will take its place.

Thus

$$\begin{aligned} P(R_1^j = 1, R_2^j = 1, R_3^j \neq 1) &= \sum_{r=2}^{n} P(R_1^j = 1, R_2^j = 1, R_3^j = r) \\ &= \sum_{r=2}^{n} \frac{1}{n^3} \\ &= \frac{n-1}{n^3}. \end{aligned}$$

Similarly,

$$P(R_1^j = 1, R_2^j \neq 1, R_3^j = 1) = \frac{n-1}{n^3} \quad \text{and} \quad P(R_1^j \neq 1, R_2^j = 1, R_3^j = 1) = \frac{n-1}{n^3}.$$

Since

$$P(R_1^j = 1, R_2^j = 1, R_3^j = 1) = \frac{1}{n^3}$$

the probability of the event that at least two committee members rank applicant j first is

$$3\frac{n-1}{n^3} + \frac{1}{n^3} = \frac{3n-2}{n^3}.$$