.

1. Proof of the Central Limit Theorem.

We want to prove the following Theorem.

Theorem 1.1. Suppose $Z_1, Z_2, \ldots, Z_n, \ldots$ is a i.i.d. sequence of random variables each with mean 0 and variance 1 and

$$S_n = \sum_{j=1}^{n} Z_j \quad \text{for each } n = 1, 2, \ldots.$$

Then

$$P(\frac{S_n}{\sqrt{n}} < z) \to \Phi(z) \quad n \to \infty \quad \text{for any } z \in \mathbb{R}.$$

(Here

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-y^2/2}\, dy \quad \text{for } z \in \mathbb{R}.)$$

Remark 1.1. The more general version we stated previously follows easily from this.

Proof. We state without proof the following Lemma.

Lemma 1.1. Suppose X_n, $n = 1, 2, \ldots$, is a sequence of random variables and Y is a random variable. Then

$$F_{X_n}(y) \to F_Y(y) \quad \text{as } n \to \infty \text{ for } y \in \mathbb{R}$$
$$\Leftrightarrow$$
$$\chi_{X_n}(t) \to \chi_Y(t) \quad \text{as } n \to \infty \text{ for } t \in \mathbb{R},$$

Remark 1.2. The really heavy lifting in the proof of the Lemma is the Fourier Inversion Theorem which we have, more or less, proved.

So we need to show that

$$\lim_{n\to\infty} \chi_{S_n/\sqrt{n}}(t) \to e^{-t^2/2} \quad \text{as } n \to \infty \text{ for } t \in \mathbb{R}. \tag{1}$$

Let $\theta : \mathbb{R} \to \mathbb{C}$ be such that $\theta(0) = 0$ and

$$e^{it} = 1 + it - \frac{t^2}{2!} + \theta(t) \quad \text{for } t \in \mathbb{R}.$$

By Taylor's Theorem with the integral form for the remainder we have

$$|\theta(t)| \leq \frac{t^3}{3!} \quad \text{for } t \in \mathbb{R}.$$

We have

$$\begin{aligned}
\chi_{S_n/\sqrt{n}}(t) &= \chi_{S_n}(t/\sqrt{n}) \\
&= \left(\chi_{X_1}(t/\sqrt{n})\right)^n \\
&= \left(E(e^{itX_1/\sqrt{n}})\right)^n \\
&= \left(1 - \frac{t^2}{2n} + E(\theta(itX_1/\sqrt{n}))\right)^n.
\end{aligned}$$

Now

$$|E(\theta(itX_1/\sqrt{n})| \leq \frac{t^3}{6n\sqrt{n}}T$$

where we have set

$$T = E(|X_1|^3).$$

Thus

$$1 - \frac{t^2}{2n} - \frac{t^3 T}{6n\sqrt{n}} \leq 1 - \frac{t^2}{2n} \leq 1 - \frac{t^2}{2n} + \frac{t^3 T}{6n\sqrt{n}}$$

for $t \in \mathbb{R}$. It follows by taking logs that

$$\left(1 - \frac{t^2}{2n} + E(\theta(itX_1/\sqrt{n}))\right)^n \to e^{-t^2/2} \quad \text{as } n \to \infty$$

for any $t \in \mathbb{R}$. □