

A very useful formula.

Let (S, \mathcal{E}, P) be a probability space.

Proposition. Suppose

(i) F_1, \dots, F_n are disjoint events with positive probability and

$$P(F_i) = P(F_j) \quad \text{whenever } i, j \in \{1, \dots, n\};$$

(ii) E is an event and

$$P(E|F_i) = P(E|F_j) \quad \text{whenever } i, j \in \{1, \dots, n\}.$$

Then

$$P(E|\cup_{i=1}^n F_i) = P(E|F_j) \quad \text{whenever } j \in \{1, \dots, n\}.$$

Proof. Suppose $j \in \{1, \dots, n\}$. We have

$$P(E \cap (\cup_{i=1}^n F_i)) = P(\cup_{i=1}^n (E \cap F_i)) = \sum_{i=1}^n P(E \cap F_i) = \sum_{i=1}^n P(E|F_i)P(F_i) = nP(E|F_j)P(F_j)$$

and

$$P(\cup_{i=1}^n F_i) = \sum_{i=1}^n P(F_i) = nP(F_j)$$

from which the desired equation immediately follows. \square

Example. Page 66, n. 2c. In bridge, what is the probability East gets 3 spades given that North and South have 8 spades between them.

Solution. Let E be the event that East gets 3 spades and let F be the event that North and South have 8 spades between them. Let

$$S = \binom{\text{Cards}}{13 \ 13 \ 13 \ 13}.$$

Evidently

$$P(\{s\}) = P(\{t\}) \quad \text{and} \quad P(E|\{s\}) = P(E|\{t\}) \quad \text{whenever } s \in F.$$

Thus, by the preceding Proposition,

$$P(E|F) = P(E|\{s\}) \quad \text{whenever } s \in F.$$

Let $s \in F$. I hope it's clear that

$$P(E|\{s\}) = \frac{\binom{5}{3} \binom{21}{10}}{\binom{26}{13}} \simeq .339.$$