

**Some worked problems from the Chapter Six homework.**

**p. 292 n. 20.** In the first case  $X$  and  $Y$  are independent because the joint density factors; in fact,

$$f_X(x) = \begin{cases} xe^{-x} & \text{if } x > 0, \\ 0 & \text{else} \end{cases} \quad \text{and} \quad f_Y(y) = \begin{cases} e^{-y} & \text{if } y > 0, \\ 0 & \text{else.} \end{cases}$$

In the second case,  $X$  and  $Y$  are not independent because the set  $\{(x, y) : 0 < x < y \text{ and } 0 < y < 1\}$  where  $f_{X,Y}$  is not zero is not a product.

**p. 292 n. 22.**

(a) No, the joint density  $f_{X,Y}$  does not factor.

(b)

$$f_X(x) = \int_0^\infty f_{X,Y}(x, y) dy = \begin{cases} \int_0^1 x + y dy & \text{if } 0 < x < 1, \\ 0 & \text{else} \end{cases} = \begin{cases} x + \frac{1}{2} & \text{if } 0 < x < 1, \\ 0 & \text{else.} \end{cases}$$

(c)

$$P(X + Y < 1) = \int \int_{x+y<1} f_{X,Y}(x, y) dx dy = \int_0^1 \left( \int_0^{1-x} (x+y) dy \right) dx = \frac{1}{3}.$$

**p. 292 n. 23.**

(a) Yes, because the joint density  $f_{X,Y}$  factors; in fact,  $f_{X,Y} = f_X(x)f_Y(y)$  where

$$f_X(x) = \begin{cases} 6x(1-x) & \text{if } 0 < x < 1, \\ 0 & \text{else,} \end{cases} \quad \text{and} \quad f_Y(y) = \begin{cases} 2y & \text{if } 0 < y < 1, \\ 0 & \text{else.} \end{cases}$$

(b)

$$E(X) = \int \int x f_{X,Y}(x, y) dx dy = \int_0^1 \left( \int_0^1 12x^2(1-x)y dy \right) dx.$$

(c)

$$E(Y) = \int \int y f_{X,Y}(x, y) dx dy = \int_0^1 \left( \int_0^1 12x(1-x)y^2 dy \right) dx.$$

(d)

$$E(X^2) = \int \int x^2 f_{X,Y}(x, y) dx dy = \dots$$

(e)

$$E(Y^2) = \int \int y^2 f_{X,Y}(x, y) dx dy = \dots$$

**p. 293 n. 27. Part One.** Let  $Z = X + Y$ .

Since the range of  $Z$  is  $(0, \infty)$  we have  $f_Z(z) = 0$  if  $z \leq 0$ . Suppose  $0 < z < \infty$ . Then, as  $X$  and  $Y$  are independent, we have

$$\begin{aligned} f_Z(z) &= f_X * f_Y(z) \\ &= \int_{-\infty}^{\infty} f_X(z-w) f_Y(w) dw \\ &= \int_{z-1}^z f_Y(w) dw \quad \text{because } 0 < z-w < 1 \Leftrightarrow z-1 < w < z \\ &= \begin{cases} \int_0^z e^{-w} dw & \text{if } z < 1, \\ \int_{z-1}^z e^{-w} dw & \text{if } z \geq 1 \end{cases} \\ &= \begin{cases} 1 - e^{-z} & \text{if } z < 1, \\ e^{-(z-1)} - e^{-z} & \text{if } z \geq 1. \end{cases} \end{aligned}$$

Integrate to get  $F_Z$  if you like.

**Part Two.** Suppose  $Z = X/Y$ . We have

$$f_{X,Y}(x,y) = f_X(x)f_Y(y) = \begin{cases} e^{-y} & \text{if } 0 < x < 1 \text{ and } 0 < y, \\ 0 & \text{else.} \end{cases}$$

Note that the range of  $Z$  is  $(0, \infty)$ . Suppose  $0 < z < \infty$ . Then

$$\begin{aligned} F_Z(z) &= P(Z \leq z) \\ &= P(X \leq zY) \\ &= \int \int_{x \leq zy} f_{X,Y}(x,y) dx dy \\ &= \int \int_{0 < x < 1, 0 < y, x \leq zy} e^{-y} dx dy \\ &= \int_0^1 \left( \int_{x/z}^{\infty} e^{-y} dy \right) dx \\ &= z(1 - e^{-1/z}). \end{aligned}$$

**p. 293, n. 28.** We have

$$f_{X_1, X_2}(x_1, x_2) = f_{X_1}(x_1)f_{X_2}(x_2) = \begin{cases} \lambda_1 \lambda_2 e^{-\lambda_1 x_1} e^{-\lambda_2 x_2} & \text{if } x_1 > 0 \text{ and } x_2 > 0, \\ 0 & \text{else.} \end{cases}$$

Let  $Z = X_1/X_2$ . Note that the range of  $Z$  is  $(0, \infty)$  so  $F_Z(z) = 0$  if  $z \leq 0$ . Suppose  $0 < z < \infty$ . Then

$$\begin{aligned} F_Z(z) &= P(Z \leq z) \\ &= P(X_1/z \leq X_2) \\ &= \int \int_{x_1/z \leq x_2} f_{X_1, X_2} dx_1 dx_2 \\ &= \lambda_1 \lambda_2 \int_0^{\infty} \left( \int_{x_1/z}^{\infty} e^{-\lambda_1 x_1} e^{-\lambda_2 x_2} dx_2 \right) dx_1 \\ &= \frac{\lambda_1 z}{\lambda_1 z + \lambda_2}. \end{aligned}$$

In particular,

$$P(X_1 < X_2) = P(Z < 1) = \frac{\lambda_1}{\lambda_1 + \lambda_2}.$$

**p. 294 n. 42.** (a) We have  $f_X(x) = 0$  if  $x < 0$ . If  $x > 0$  we have

$$f_X(x) = \int_0^\infty f_{X,Y}(x,y) dy = xe^{-x} \int_0^\infty e^{-xy} dy = xe^{-x} \frac{1}{x} = e^{-x}.$$

Thus if  $x > 0$  (so  $f_X(x) > 0$ ) we have

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \begin{cases} xe^{-xy} & \text{if } y > 0, \\ 0 & \text{else.} \end{cases}$$

We have  $f_Y(y) = 0$  if  $y < 0$ . If  $y > 0$  we have

$$f_Y(y) = \int_0^\infty xe^{-x(y+1)} dx = - \int_0^\infty x d_x \left( \frac{e^{-x(y+1)}}{y+1} \right) = -x \frac{e^{-x(y+1)}}{y+1} \Big|_{x=0}^{x=\infty} + \int_0^\infty \frac{e^{-x(y+1)}}{y+1} dx = \frac{1}{(y+1)^2}.$$

Thus if  $y > 0$  (so  $f_Y(y) > 0$ ) we have

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \begin{cases} \frac{x}{(y+1)^2} e^{-x(y+1)} & \text{if } x > 0, \\ 0 & \text{else.} \end{cases}$$

(b) Suppose  $z > 0$  and  $y > 0$ . Then

$$f_{XY|Y}(z|y) = f_{yX|Y}(z|y) = \frac{1}{y} f_{X|Y}\left(\frac{z}{y}|y\right) = \frac{1}{y} \frac{f_{X,Y}\left(\frac{z}{y}, y\right)}{f_Y(y)}.$$

Therefore,

$$f_{XY}(z) = \int_0^\infty f_{XY|Y}(z|y) f_Y(y) dy = \int_0^\infty \frac{1}{y} \frac{z}{y} e^{-\frac{z}{y}(y+1)} dy = ze^{-z} \int_0^\infty \frac{e^{-\frac{z}{y}}}{y^2} dy = e^{-z} \int_0^\infty \frac{e^{-\frac{1}{w}}}{w^2} dw = e^{-z}$$

where we have made the substitution  $y = zw$  and deduced that  $\int_0^\infty \frac{e^{-\frac{1}{w}}}{w^2} dw = 1$  since  $\int_0^\infty f_{XY}(z) dz$  must be 1 and since  $\int_0^\infty e^{-z} dz = 1$ .

**p. 294 n. 43.** We have  $f_X(x) = 0$  for  $x < 0$ . For  $x > 0$  we have

$$f_X(x) = \int_0^\infty f_{X,Y}(x,y) dy = c \int_{-x}^x (x^2 - y^2) e^{-x} dy = \frac{4c}{3} x^3 e^{-x}.$$

Thus, for  $x > 0$  we have

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \begin{cases} \frac{3}{4} \frac{x^2 - y^2}{x^3} & \text{if } -x < y < x, \\ 0 & \text{else.} \end{cases}$$

Finally,  $F_{Y|X}(y|x) = 0$  if  $y \leq -x$ ,  $F_{Y|X}(y|x) = 1$  if  $y \geq x$  and, if  $-x < y < x$ ,

$$F_{Y|X}(y|x) = \int_{-x}^y \frac{3}{4} \frac{x^3 - w^3}{x^3} dw = \frac{1}{2} + \frac{3y}{4x} - \frac{1}{4x^3}.$$