Some worked problems from Chapter Five.

Page 228, n. 2. A system consisting of one original unit plus a spare can function for a random amount of time X. If the density of X is given (in units of months) by

$$f_X(x) = C \begin{cases} xe^{-x/2}\,dx & \text{if } x > 0; \\ 0 & \text{else} \end{cases}$$

what is the probability the system functions for at least 5 months.

Solution. X must be exponential so C is $\frac{1}{2}$ and

$$P(X \geq 5) = e^{-\frac{5}{2}} \simeq .082.$$

Page 229, n. 4. Suppose X is uniform on $(0, 1)$ and n is a positive integer. Compute $E(X)$ in two different ways.

Note that

$$f_X(x) = \begin{cases} 0 & \text{if } x \leq 0; \\ 1 & \text{if } 0 \leq x < 1; \\ 0 & \text{if } 1 \leq x. \end{cases}$$

Solution One. Applying the formula that says

$$E(g(X)) = \int_{-\infty}^{\infty} g(x)\, f_X(x)\, dx$$

we obtain

$$E(X) = \int_0^1 x^n\, f(x)\, dx = \frac{x^{n+1}}{n+1}\Big|_{x=0}^{x=1} = \frac{1}{n+1}.$$

Solution Two. Let $Y = X^n$. Then the range of Y is $(0, 1)$ and

$$F_Y(y) = P(Y \leq y) = P(X^n \leq y) = P(X \leq y^{\frac{1}{n}}) = \int_0^{y^{\frac{1}{n}}} 1\, dx = y^{\frac{1}{n}}$$

so

$$f_Y(y) = \begin{cases} 0 & \text{if } y \leq 0; \\ \frac{1}{n} y^{\frac{1}{n}-1} & \text{if } 0 < y \leq 1; \\ 0 & \text{if } 1 \leq y. \end{cases}$$

Thus

$$E(Y) = \int_0^1 y\, f_Y(y)\, dy = \int_0^1 \frac{1}{n} y^{\frac{1}{n}}\, dy = \frac{1}{n} \frac{y^{\frac{1}{n}+1}}{\frac{1}{n}+1}\Big|_0^1 = \frac{1}{n} \frac{1}{\frac{1}{n}+1} = \frac{1}{n+1}.$$

Page 229, n. 16. The annual rainfall (in inches) in a certain region is normally distributed with $\mu = 40$ and $\sigma = 4$. What is the probability that starting with this year, it will take over 10 years before a year occurs having a rainfall of over 50 inches. What assumptions are you making?

Solution. We assume that the rainfall in the various years are mutually independent. Let X be geometric with parameter

$$p = P(Y \geq 50)$$

where Y is normal with mean 40 and variance 4. Since $\frac{Y-40}{4}$ is normal with mean 0 and variance 1 we obtain from the table on page 203 that

$$P(Y \geq 50) = P\Big(\frac{Y-40}{4} \geq \frac{10}{4}\Big) = 1 - P\Big(\frac{Y-40}{4} < \frac{10}{4}\Big) = 1 - \Phi(\frac{10}{4}) \simeq 1 - .9938 = .0062.$$

Thus, with $q = 1 - p \simeq .9938$, we find that the answer to the problem is $P(X \geq 10) \simeq q^10$.

Page 230, n. 17. A main aiming at a target receives 10 points if his shot is within 1 inch of the target, 5 points if it is between 1 and 3 inches of the target, and 3 points if it between 3 and 5 inches of the target. Find the expected number of points scored if the distance from the shot to the target is uniformly distributed between 0 and 10.

Solution. Let S be uniform on $(0, 10)$ and let X be the number of points scored. The

$$P(X = 10) = P(S \leq 1) = \frac{1-0}{10};$$
$$P(X = 5) = P(1 < S \leq 3) = \frac{3-1}{10};$$
$$P(X = 3) = P(3 < S \leq 5).$$

Thus

$$E(X) = 10 \cdot \frac{1}{10} + 5 \cdot \frac{2}{10} + 3 \cdot \frac{2}{10} = \frac{26}{10}.$$

Page 230, n. 19. let X be a normal random variable with mean 12 and variance 4. Find the value of c such that $P(X > c) = .10$.

Solution. Since c is larger than the mean 12 (Why?) and since $\frac{X-12}{2}$ is normal with mean 0 and variance 1 we have

$$P(X > c) = P\Big(\frac{X-12}{2} > \frac{c-12}{2}\Big) = 1 - P\Big(\frac{X-12}{2} \leq \frac{c-12}{2}\Big) = 1 - \Phi(\frac{c-12}{2})$$

so $\Phi(\frac{c-12}{2}) = .9$ so, from the table on page 203, $\frac{c-12}{2} \simeq 1.28$ so $c \simeq 14.56$.

Page 230, n. 24. The lifetimes of interactive computer chips produced by a certain semiconductor manufacturer are normally distributed with parameters $\mu = 1.4 \times 06^6$ hours and $\sigma = 3 \times 10^6$ hours. What is the approximate probability that a batch of 100 chips will contain at least 20 whose lifetimes are less that 1.8×10^6?

Solution. Suppose X_i is the lifetime of the i-th chip, $i = 1, 2 \ldots, 100$. We assume the X_i are independent. Let

$$p = P(X_i \leq 1.8 \times 10^6).$$

Since $\frac{X_i - 1.4^6}{3 \times 10^6}$ is normally distributed with mean 0 and variance 1 we obtain from the table on page 203 that

$$p = P\Big(\frac{X_i - 1.4^6}{3 \times 10^6} \leq \frac{.4 \times 10^6}{3 \times 10^6}\Big) = \Phi(\frac{4}{3}) \simeq .5517.$$

For each $i = 1, 2, \ldots, 100$ let

$$Y_i = \begin{cases} 1 & \text{if } X_i \le 1.8 \times 10^6, \\ 0 & \text{else.} \end{cases}$$

Then $S = \sum_{i=1}^{100} Y_i$ is binomial with parameters 100 and p; thus, with $q = 1 - p$, we use the normal approximation to the binomial and the table on page 203 to obtain

$$P(S \ge 20) = P\Big(\frac{S - 100p}{\sqrt{100pq}} \ge \frac{20 - 100p}{\sqrt{100pq}} \Big) \simeq 1.$$

Page 231, n. 31(a). A fire station is to be located along a road of length A, $A < \infty$. If fires will occur at points uniformly chosen on (p, A), where should the station be located so as to minimize the expected instance from the fire? That is, choose a so as to minimize $E(|X - a|)$ where X is uniformly distributed on $(0, A)$.

Solution. Since

$$f_X(x) = \frac{1}{A} \begin{cases} 0 & \text{if } x \le 0; \\ 1 & \text{if } 0 < x \le A; \\ 0 & \text{if } A < x \end{cases}$$

we have

$$E(|X - a|) = \int_{-\infty}^{\infty} |x - a| f_X(x)\, dx = \frac{1}{A} \int_0^A |x - a|\, dx = \frac{1}{A} a^2 + (A - a)^2$$

which is easily seen to have its unique minimum value on $(0, A)$ at $a = \frac{A}{2}$. This makes sense.

Problem 9 on page 229. Introduction. Suppose Y is a random variable and $t \in \mathbf{R}$. Let

$$Z = \min\{Y, t\}.$$

Let's compute F_Z in terms of F_Y. For any $z \in \mathbf{R}$ we have

$$Z \le z \ \Leftrightarrow\ Y \le z \text{ and } t \le z$$

from which it follows that

$$F_Z(z) = P(Z \le z) = \begin{cases} P(Y \le z) & \text{if } z < t, \\ 1 & \text{if } z \ge t \end{cases} = \begin{cases} F_Y(z) & \text{if } z < t, \\ 1 & \text{if } z \ge t. \end{cases}$$

Suppose

$$F_Y(t) < 1.$$

Then

$$\lim_{z \uparrow t} F_Z(z) = \lim_{z \uparrow t} F_Y(z) \le F_Y(t) < 1 = F_Z(t)$$

so F_Z is discontinous at t. As the cdf of a continuous random variable is always continuous (Why?) we find that Z will never be continuous. If Y is continuous, t will be the only point at which F_Z is discontinuous.

Now let us assume that Y is continuous and compute the expectation of Z. Using a fundamental formula for expectation we find that

$$(1) \quad E(Z) = \int_{-\infty}^{\infty} \min\{y, t\} f_Y(y)\, dy = \int_{-\infty}^{t} y f_Y(y)\, dy + \int_t^{\infty} t f_Y(y)\, dy = \int_{-\infty}^{t} y f_Y(y)\, dy + t\,[1 - F_Y(t)]\,.$$

The essence of p.229, n. 9. You sell widgets which you buy from the wholesaler at a cost of c dollars per widget and which you sell for $c+p$ dollars each. Since the wholesaler neither gives away widgets nor pay you to take them, c is positive. Since you are good at business, you profit p per widget sold is positive. Now the wholesaler will only sell you widgets at the beginning of the week and he will not take back any unsold widgets because, as everyone knows, widgets go bad in exactly one week. Let X be the (random) demand for widgets in a week and let s be the number of widgets you stock, that is, buy from the wholesaler at the beginning of the week. We assume X is continuous; evidently, $X \geq 0$. Your goal is to determine s so that the expectation of your profit

$$P_s = (p+c)\min\{X, s\} - cs$$

is maximized; $\min\{X, s\}$ is the number you sell because you only have s widgets to sell.

In view of the above we have

$$E(P_s) = (p+c)\left(\int_{-\infty}^{s} x f_X(x)\,dx + s[1 - F_X(s)]\right) - cs. \tag{1}$$

Since F_X is continuous, $\lim_{x\downarrow-\infty} F_X(x) = 0$, $\lim_{c\uparrow\infty} F_X(x) = 1$ and $0 < \frac{c}{p+c} < 1$ there is s^* such that

$$F_X(s^*) = \frac{c}{p+c}.$$

This gives

$$(p+c)E(P_s) = \int_{-\infty}^{s} x f_X(x)\,dx + s[F_X(s^*) - F_X(s)].$$

I claim that

$$E(P_s) \leq E(P_{s^*}) \quad \text{for any } s.$$

If $s < s^*$ we have

$$\begin{aligned}
(p+c)E(P_s) &= \int_{-\infty}^{s} x f_X(x)\,dx + s[F_X(s^*) - F_X(s)] \\
&= \int_{-\infty}^{s} x f_X(x)\,dx + s\int_{s}^{s^*} f_X(x)\,dx \\
&\leq \int_{-\infty}^{s} x f_X(x)\,dx + \int_{s}^{s^*} x f_X(x)\,dx \\
&= \int_{-\infty}^{s^*} x f_X(x)\,dx = (p+c)E(P_{s^*}).
\end{aligned}$$

If $s^* < s$ we have

$$\begin{aligned}
(p+c)E(P_s) &= \int_{-\infty}^{s} x f_X(x)\,dx + s[F_X(s^*) - F_X(s)] \\
&= \int_{-\infty}^{s^*} x f_X(x)\,dx + \int_{s^*}^{s} x f_X(x)\,dx - s\int_{s^*}^{s} f_X(x)\,dx \\
&= \int_{-\infty}^{s^*} x f_X(x)\,dx + \int_{s^*}^{s} (x-s) f_X(x)\,dx \\
&= \int_{-\infty}^{s^*} x f_X(x)\,dx = (p+c)E(P_{s^*}).
\end{aligned}$$