Definition. Suppose $0 \le p \le 1$ and n is a positive integer. We say a random variable X has the **binomial distribution**

$$b(n, p)$$

(see p.63) if

$$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k} \quad \text{for } i = 0, 1, \ldots, n.$$

Discussion. Consider an experiment $\mathcal{E}$ which has two outcomes: success with probability p and failure with probability $1-p$. Let n be a positive integer. Let $\mathcal{E}_n$ be the experiment obtained by performing $\mathcal{E}$ n times in succession. A sample space for $\mathcal{E}_n$ can be

$$S = \{s : s \subset \{0, 1, \ldots, n\}\}$$

if we think of $s \in S$ as being the set of thoses indices $i = 1, \ldots, n$ which correspond to success. For each $i = 1, \ldots, n$ we have random variables

$$X_i : S \longrightarrow \{0, 1\}$$

defined by

$$X_i(s) = \begin{cases} 1 & \text{if } i \in s \\ 0 & \text{else} \end{cases} \quad \text{for } s \in S.$$

Thus $P(X_i = 1) = p$ and $P(X_i = 0) = 1 - p$ and the events

$$\{X_i = a_i\}, \quad i = 1, \ldots, n$$

are independent no matter what the a_i's are. Let

$$X = \sum_{i=1}^{n} X_i.$$

Note that

$$X(s) = |s| \quad \text{for } s \in S$$

which is to say that X is the *number of successes in n trials.* For any $s \in S$ we have

$$\{s\} = \bigcap_{i \in s} \{X_i = 1\} \cap \bigcap_{i \notin S} \{X_i = 0\}$$

so, by independence,

$$P(\{s\}) = p^{|s|}(1-p)^{n-|s|} = p^{X(s)}(1-p)^{n-X(s)}$$

for $s \in S$. Thus, for any $k \in \{0, 1, \ldots, n\}$,

$$P(X = k) = \sum_{\{s \in S : |s| = k\}} p^k (1-p)^{n-k} = \binom{n}{k} p^k (1-p)^{n-k}$$

Thus X has the binomial distribution $b(n, p)$.