Bayes' Rule. Suppose $B_1, \ldots, B_n$ are disjoint events each of which with positive probability and whose union is the entire sample space. Then for any event A and any $j \in \{1, \ldots, n\}$ we have

$$P(B_i|A) = \frac{P(A|B_j)P(B_j)}{\sum_{i=1}^{n} P(A|B_i)P(B_i)}.$$

Proof. We have already shown that the denominator of the right hand side equals $P(A)$. Then numerator equals $P(A \cap B_i)$. $\square$

So you don't get prize for proving Bayes' Rule. It's interesting because of the way it can be applied.

Example 3d, page 72. A laboratory blood test is 95 percent effective in detecting a certain disease when it is, in fact, present. However, the test also yields a "false positive" result for 1 percent of the healthy persons tested. If .5 percent of the population actually has the disease, what is the probability that a person has the disease given that the test is positive?

Solution. Let B_1 be the event that a person has the disease, let B_2 be the event that a person does not have the disease and let A be the event that the test result is positive. Then

$$P(B_1) = .005, \quad P(B_2) = 1 - P(B_1) = .995,$$

$$P(A|B_1) = .95, \quad P(A|B_2) = .01$$

so

$$P(B_1|A) = \frac{P(A|B_1)P(B_1)}{P(A|B_1)P(B_1) + P(A|B_2)P(B_2)} = \frac{(.95)(.005)}{(.95)(.005) + (.01)(.995)} = \frac{95}{294} \simeq .323.$$

More generally, we could set

$$p = P(B_1), \quad R = \frac{P(A|B_2)}{P(A|B_1)}$$

and obtain

$$P(B_1|A) = \frac{p}{p + R(1-p)}.$$

Think of R as *badness ratio*; the higher R the worse the test is. The badness ration of this test is

$$\frac{.01}{.95} = .0105.$$