Probem 7(a) on page 116.

A paraphrase of the problem. An urn contains a finite nonempty set B of black balls and a finite nonempty set W of white balls and nothing else. Balls are drawn from the urn without replacement until the remaining balls are of the same color. What is the probability that the remaining ball(s) are white?

Solution. Let $A = B \cup W$ and let $M = \{1, \dots, |A| - 1\}$. For each $m \in M$ let

$$E_m$$

be the set of $x \in (A)_m$ such that

$$x_m \in B \quad \text{and} \quad |\{i \in \{1, \dots, m\} : x_i \in B\}| = |B|;$$

let

$$F_m$$

be the set of $x \in (A)_m$ such that

$$x_m \in W \quad \text{and} \quad |\{i \in \{1, \dots, m\} : x_i \in W\}| = |W|.$$

and let

$$S_m = E_m \cup F_m.$$

Note that whenever $m \in M$ we have $E_m = \emptyset$ if $m < |B|$ and $F_m = \emptyset$ if $m < |W|$. Evidently, the family $\{S_m : m \in M\}$ is disjointed and

$$S = \cup_{m=1}^{|A|-1} S_m$$

is a reasonable sample space for this experiment. For any $m \in M$, E_m is the event the that the m-th ball drawn is black and that all the black balls have been drawn and F_m is the event that the m-th ball drawn is white and that all the white balls have been drawn.

Moreover,

$$P(\{s\}) = \frac{1}{(|A|)_m} \quad \text{whenever } m \in M\} \text{ and } s \in S_m;$$

this is because on a given draw the probability of getting a given ball still in the urn on a given draw is one over the number of balls in the urn. Let

$$E = \cup_{m=|B|}^{|A|-1} E_m.$$

We need to calculate $P(E)$.

Let

$$R = (A)_{|A|}.$$

For each $m \in M$ we let

$$C_m$$

be the set of those $r \in R$ such that

$$r_m \in B \quad \text{and} \quad r_i \in W \quad \text{whenever } i \in \{m+1, \dots, |A|\};$$

we let

$$D_m$$

be the set of those $r \in R$ such that

$$r_m \in W \quad \text{and} \quad r_i \in B \quad \text{whenever} i \in \{m+1, \dots, |A|\};$$

and we let

$$R_m = C_m \cup D_m.$$

Note that if then $C_m = \emptyset$ if $m < |B|$ and $D_m = \emptyset$ if $m < |W|$. Evidently, R is the disjoint union of the family $\{R_1, \ldots, R_M\}$. We may define

$$f : R \to S$$

by requiring that

$$f(r) = (r_1, \ldots, r_m) \quad \text{whenever } m \in M \text{ and } r \in R_m.$$

Note that $f[C_m] = E_m$ and $f[D_m] = F_m$ for any $m \in M$. Moreover,

$$|\{r \in R : f(r) = s\}| = |\{r \in R_m : f(r) = s\}| = (|A| - m)! \quad \text{whenvever } m \in M \text{ and } r \in R_m..$$

Thus

$$P(E_m) = \frac{|E_m|}{(|A|)_m} = \frac{|F_m|}{(|A| - m)!(|A|)_m} = \frac{|F_m|}{|A|!} \quad \text{whenever } m \in M.$$

It follows that

$$P(E_m) = \sum_{m=|B|}^{|A|-1} \frac{|F_m|}{|A|!} = \frac{1}{|A|!} | \cup_{m=|B|}^{|A|-1} F_m| = \frac{|W|}{|B| + |W|}$$

because

$$| \cup_{m=|B|}^{|A|-1} F_m| = |\{r \in (A)_|A| : r_{|A|} \in W\}| = (|A| - 1)!|W|.$$