Variation of Constants?

Let us recall how we solved

$$\text{(IVP).}\qquad \frac{dy}{dx} + p(x)y(x) = q(x), \quad y(x_0) = y_0$$

First solve

$$\frac{d\mu}{dx} + p(x)y(x) = 0, \quad \mu(x_0) = 1.$$

(This μ is the reciprocal of the one we considered earlier, but it fits the current context better as we shall see.) We calculate

$$\begin{aligned}\frac{d}{dx}\left(\mu(x)^{-1}y(x)\right) &= -\mu(x)^{-2}\frac{d\mu}{dx}y(x) + \mu(x)^{-1}\frac{dy}{dx}\\ &= -\mu(x)^{-2}\left(-p(x)\mu(x)\right) + \mu(x)^{-1}\left(-p(x)y(x) + q(x)\right)\\ &= \mu(x)^{-1}q(x)\end{aligned}$$

so that

$$y(x) = \mu(x)\left(y_0 + \int_{x_0}^{x} \mu(\xi)^{-1}q(\xi)\,d\xi\right).$$

Now let n be a positive integer greater than one and consider the *system*

$$\text{(IVP)}\qquad \frac{dY}{dx} + P(x)Y(x) = Q(x), \quad Y(x_0) = Y_0$$

where

$$Y(x) = \begin{bmatrix} Y_1(x)\\ \vdots \\ Y_n(x)\end{bmatrix}, \quad P(x) = \begin{bmatrix} p_{11}(x) & \cdots & p_{1n}(x)\\ \vdots & \ddots & \vdots\\ p_{n1}(x) & \cdots & p_{nn}(x)\end{bmatrix}, \quad Q(x) = \begin{bmatrix} Q_1(x)\\ \vdots \\ Q_n(x)\end{bmatrix}, \quad Y_0 = \begin{bmatrix} y_1\\ \vdots \\ y_n\end{bmatrix}.$$

Note that Y is a column vector of functions; P is a (square) matrix of functions; Q is a column vector of functions; and Y_0 is a column vector of constants. Let

$$\mu_1 = \begin{bmatrix} \mu_{11}\\ \vdots \\ \mu_{n1}\end{bmatrix}, \ldots, \mu_n \begin{bmatrix} \mu_{1n}\\ \vdots \\ \mu_{nn}\end{bmatrix}$$

be the (column) vectors of functions that solve the IVP's

$$\frac{d\mu_j}{dx} + P(x)\mu_j(x) = 0, \quad \mu_j(x_0) = \mathbf{e}_j,\ j = 1, \ldots, n,$$

and let

$$\mu(x) = [\,\mu_1(x) \quad \cdots \quad \mu_n(x)\,];$$

here $\mathbf{e}_j$, $j = 1, \ldots, n$, the **j-th standard basis vector** in $\mathbf{R}^n$, has 1 in the j-th row and zeroes in the other rows. This μ is called a **fundamental solution of $Ly = 0$.** Note that μ is an $n \times n$ matrix of functions and that

$$\frac{d\mu}{dx} + P(x)\mu(x) = 0, \quad \mu(x_0) = I$$

where I is the $n \times n$ identity matrix.

Example. Say $n = 2$ and

$$P(x) = -\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Then

$$[\,\mu_1(x) \quad \mu_2(x)\,] = \begin{bmatrix} \mu_{11}(x) & \mu_{12}(x) \\ \mu_{21}(x) & \mu_{22}(x) \end{bmatrix} = \begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix}.$$

We have

$$\begin{aligned} \frac{d}{dx}\left(\mu(x)^{-1}Y(x)\right) &= -\mu(x)^{-1}\frac{d\mu}{dx}\mu(x)^{-1}Y(x) + \mu(x)^{-1}\frac{dY}{dx} \\ &= -\mu(x)^{-1}\left(-P(x)\mu(x)\right)\mu(x)^{-1}Y(x) + \mu(x)^{-1}\left(-P(x)Y(x) + Q(x)\right) \\ &= \mu(x)^{-1}Q(x) \end{aligned}$$

so

$$\mu(x)^{-1}Y(x) - Y(x_0) = \int_{x_0}^{x} \mu(\xi)^{-1}Q(\xi)\,d\xi$$

or

$$Y(x) = \mu(x)\left(Y_0 + \int_{x_0}^{x} \mu(\xi)^{-1}Q(\xi)\,d\xi\right).$$

Note that if we let vector in parentheses equal

$$\begin{bmatrix} v_1(x) \\ \vdots \\ v_n(x) \end{bmatrix}$$

then

$$\text{(VofP)} \qquad Y(x) = v_1(x)\mu_1(x) + \cdots + v_n(x)\mu_n(x).$$

Thus if the μ's are "constants" then Y is obtained by "varying" the constants with the v's; that's why (I think) they call this method of obtaining a solution of the inhomogeneous problem "variation of constants".

Now consider the LDO

$$Ly(x) = y^{(n)}(x) + a_{n-1}y^{(n-1)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y(x)$$

and consider

$$\text{(IVP}'\text{)} \qquad Ly(x) = q(x), \quad y(x_0) = y_0, \ldots, y^{(n-1)}(x_0) = y_{n-1}$$

where q is a given function and $y_0, \ldots, y_{n-1}$ are given constants. Let

$$Y(x) = \begin{bmatrix} y(x) \\ y'(x) \\ \vdots \\ y^{(n-2)}(x) \\ y^{(n-1)}(x) \end{bmatrix}, \quad P(x) = -\begin{bmatrix} 0 & 1 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ a_0(x) & a_1(x) & \ldots & a_{n-2}(x) & a_{n-1}(x) \end{bmatrix}, \quad Q(x) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ q(x) \end{bmatrix}$$

and

$$Y_0 = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-2} \\ y_{n-1} \end{bmatrix}.$$

Then IVP$'$ is equivalent to IVP. Moreover, VofP becomes

$$y(x) = v_1(x)\mu_{11}(x) + \cdots + v_n(x)\mu_{1n}(x).$$

It customary to let the **Wronskian**

$$W(x) = \mathbf{det}\, \mu(x),$$

and to let

$$M(x)$$

be the cofactor matrix of $\mu(x)$ so that

$$\mu(x)^{-1} = \frac{1}{W(x)} M(x).$$

In case $n = 2$, for example, if y_1, y_2 are the solutions of $Ly = 0$ such that

$$\begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(note the clash of notations!) then

$$\mu(x) = \begin{bmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{bmatrix}$$

and we have

$$W(x) = y_1(x)y_2'(x) - y_2(x)y'(x)$$

and

$$M(x) = \begin{bmatrix} y_2'(x) & -y_2(x) \\ -y_1'(x) & y_1(x) \end{bmatrix}.$$