

Time to blowup.

Suppose J is an open interval in \mathbf{R} and

$$f : J \rightarrow \mathbf{R}$$

is continuously differentiable and consider

$$(ODE) \quad x'(t) = f(x(t)).$$

Suppose $x_0 \in \mathbf{R}$ and

$$(1) \quad f(x) > 0 \quad \text{whenever } x \geq x_0.$$

Let $x : I \rightarrow \mathbf{R}$ be the maximal solution of (ODE) such that $x(0) = x_0$.

Proposition. We have

$$(2) \quad x'(t) > 0 \quad \text{whenever } t \in I \text{ and } 0 \leq t.$$

Proof. Extra credit exercise. \square

Let

$$T = \sup I;$$

since $0 \in I$ by the definition of maximal solution we find that

$$0 < T \leq \infty.$$

It follows from (2) that

$$X = \lim_{t \uparrow T} x(t) \text{ exists.}$$

Proposition. We have

$$(3) \quad T < \infty \Rightarrow X = \sup J.$$

Proof. Extra credit exercise. \square

I claim that

$$X = \sup B$$

and that

$$(4) \quad T = \int_{x_0}^{\sup J} \frac{dx}{f(x)}.$$

We need a basic formula from calculus.

Proposition. Suppose $-\infty < c < d < \infty$;

$$g : [c, d] \rightarrow \mathbf{R};$$

g is continuous; $-\infty < a < b < \infty$;

$$\phi : [a, b] \rightarrow [c, d];$$

ϕ is continuous; and ϕ is continuously differentiable on (c, d) ; and

$$\phi(a) = c \text{ and } \phi(b) = d.$$

Then

$$\int_c^d g(y) dy = \int_a^b g(\phi(x))\phi'(x) dx.$$

Proof. Let $G : [c, d] \rightarrow \mathbf{R}$ be such that G is continuous on $[c, d]$ and $G'(y) = g(y)$ whenever $y \in (c, d)$. (For example, we could set

$$G(y) = \int_c^y g(\zeta) d\zeta, \quad y \in [c, d].)$$

Using the chain rule and the fundamental theorem of calculus we calculate

$$\begin{aligned} & \int_a^b g(\phi(x))\phi'(x) dx \\ &= \int_a^b (G \circ \phi)'(x) dx \\ &= G(\phi(b)) - G(\phi(a)) \\ &= G(d) - G(c) \\ &= \int_c^d g(y) dy. \end{aligned}$$

□

Suppose $0 < t < T$. Using the preceding Proposition with a, b, c, d, ϕ there equal $0, t, x_0, x(t), x$ we calculate

$$\begin{aligned} (5) \quad t &= \int_0^t 1 d\tau \\ &= \int_0^t \frac{1}{f(x(\tau))} x'(\tau) d\tau \\ &= \int_{x_0}^{x(t)} \frac{d\xi}{f(\xi)}. \end{aligned}$$

Letting $t \uparrow T$ in (4) we obtain

$$(6) \quad T = \int_0^X \frac{d\xi}{f(\xi)}.$$

Were it the case that $X < \sup J$ we could infer from (6) that $T < \infty$ and that would contradict (3).