Some fixed point theorems.

Suppose X is nonempty set and $<$ is a complete linear ordering of X. Given $x, y \in X$ we write

$$x \leq y$$

if $x < y$ or $x = y$. Whenever $a, b \in X$ and $a < b$ we let

$$(a, b) = \{x \in X : a < x < b\}$$

and we assume that

$$(a, b) \neq \emptyset. \tag{1}$$

We let $\mathcal{F}$ be the set of f such that

(i) $f : X \to X$;

(ii) if $x, y \in X$, $x \leq y$ then $f(x) \leq f(y)$;

(iii) if $x, y \in X$, $x < y$, $w \in X$ and $f(x) < w < f(y)$ there is $v \in X$ such that

$$x < v < y \quad \text{and} \quad w = f(v).$$

We let $\mathcal{G}$ be the set of g such that

(iv) $g : X \to X$;

(v) if $x, y \in X$ and $x \leq y$ then $g(y) \leq g(x)$;

(vi) if $x, y \in X$, $x < y$, $t \in X$ and $g(y) < w < g(x)$ there is $v \in X$ such that

$$x < v < y \quad \text{and} \quad w = f(v).$$

Theorem. Suppose S is a nonempty subset of X and $f \in \mathcal{F}$. If S has an upper bound then $f[S]$ has an upper bound and

$$f(\sup S) = \sup f[S].$$

If S has a lower bound then $f[S]$ has a lower bound and

$$f(\inf S) = \inf f[S].$$

Proof. Suppose b is and upper bound for S. If $s \in S$ then $s \leq b$ so $f(s) \leq f(b)$ so $f(b)$ is an upper bound for S. In particular,

$$f(s) \leq f(\sup S) \quad \text{whenever } s \in S. \tag{2}$$

It follows directly from (2) that $\sup f[S] \leq f(\sup S)$. Suppose, contrary to the Theorem, $\sup f[S] < f(\sup S)$. Keeping in mind (1) we choose $w \in X$ such that $\sup f[S] < w < f(\sup S)$. Let $s' \in S$. Then $f(s') \leq \sup f[S] < w < f(\sup S)$. From (iii) we infer that there is $v \in X$ such that $s' < v < \sup S$ and $w = f(v)$. Since $v < \sup S$ there is $s \in S$ such that $v < s \leq \sup S$. This implies $w = f(v) \leq f(s) \leq \sup f[S]$ which is impossible.

One may prove the second assertion in a similar fashion; we leave the details to the reader. $\square$

Theorem. Suppose S is a nonempty subset of X and $g \in \mathcal{G}$. If S has an upper bound then $g[S]$ has a lower bound and

$$g(\sup S) = \inf g[S].$$

If S has a lower bound then $g[S]$ has an upper bound and

$$g(\inf S) = \sup g[S].$$

Proof. One may prove this in the same fashion as the previous Theorem was proved. We leave the details to the reader. $\square$

Theorem. Suppose $f \in \mathcal{F}$. Then *either*

$$\{f^n(x) : n \in \mathbf{N}\} \quad \text{has no upper bound whenever } x \in X$$

or

$$\text{there is } a \text{ such that } a \in X \text{ and } f(a) = a$$

Proof. Suppose $x \in X$, $S = \{f^n(x) : n \in \mathbf{N}\}$ and S has an upper bound. Note that $f[S] = S$. Thus, by a previous Theorem,

$$f(\sup S) = \sup f[S] = \sup S.$$

$\square$

Theorem. Suppose $g \in \mathcal{G}$. Then there is a such that $a \in X$ and $g(a) = a$.
Proof. Let

$$A = \{x \in X : x < g(x)\}, \quad B = \{x \in X : g(x) = x\}, \quad C = \{x \in X : g(x) < x\}$$

and note that $\{A, B, C\}$ is a partition of X.

Suppose, contrary to the Theorem, $B = \emptyset$. Whenever $x \in A$ we have $g(x) \leq x$; since C is empty we cannot have $g(x) = x$ so $g(x) \in B$; that is, $g[A] \subset B$. In a similar fashion we may infer that $g[B] \subset A$. In particular, neither A nor B is empty.

Suppose $a \in A$ and $b \in B$. Were it the case that $b < a$ we would have

$$a < g(a) \leq g(b) < b$$

which is impossible. Thus every member of A is a lower bound for B and every member of B is an upper bound for A which implies that $\sup A \leq \inf B$. Were it the case that $\sup A < \inf B$ by (1) there would be $c \in X$ such that $\sup A < c < \inf B$; but then $c \notin A$ and $c \notin B$ which is impossible. Thus

$$\sup A = \inf B.$$

Finally, we use the previous Theorem and the fact that $g[A] \subset B$ to infer that

$$\inf B \leq \inf g[A] = g(\sup A) = g(\inf B)$$

which is incompatible with $C = \emptyset$. $\square$