**Betweenness preserving permutations.**

We let

$$\iota$$

be the identity map of the set of points.

**Definition.** We say a permutation $\tau$ of the set of points is **betweenness preserving** if

$$\tau[\mathbf{s}(a,b)] = \mathbf{s}(\tau(a),\tau(b)) \text{ whenever } a \text{ and } b \text{ are distinct points.}$$

We let

$$\mathcal{B}$$

be the set of betweenness preserving permutations of the set of points.

**Theorem.** $\mathcal{B}$ is a subgroup of the group of permutations of the set of points.
**Proof.** Exercise. $\square$

**Remark.** We will use repeatedly the fact that if $\tau \in \mathcal{B}$ and $X$ is a set of points then

$$\tau^{-1}[\tau[X]] = (\tau^{-1} \circ \tau)[X] = \iota[X] = X.$$

**Proposition.** Suppose $A$ is a set of points and $\tau \in \mathcal{B}$. Then

$$\tau[\mathbf{b}(A)] = \mathbf{b}(\tau[A]).$$

**Proof.** Suppose $y \in \tau[\mathbf{b}(A)]$. Then there is $x$ in $\mathbf{b}(A)$ such that $y = \tau(x)$. Since $x \in \mathbf{b}(A)$ there is $a$ in $A$ such that $\mathbf{s}(a,x) \subset A$. Let $b = \tau(a)$ and note that $b \in \tau[A]$. Since $\tau \in \mathcal{B}$ we have $\tau[\mathbf{s}(a,x)] = \mathbf{s}(\tau(a),\tau(x)) = \mathbf{s}(b,y)$. Were it the case that $y \in \tau[A]$ we would have $x = \tau^{-1}(y) \in \tau^{-1}[\tau[A]] = A$ which is impossible since $x \in \mathbf{b}(A)$ implies $x \notin A$. Thus $y \in \mathbf{b}(\tau[A])$.

Suppose $y \in \mathbf{b}(\tau[A])$. Then $y \notin \tau[A]$ and there is $b$ in $\tau[A]$ such that $\mathbf{s}(b,y) \subset \tau[A]$. Let $x = \tau^{-1}(y)$ and let $a = \tau^{-1}(b)$. Were it the case that $x \in A$ we would have $y = \tau(x) \in \tau[A]$ which is impossible. Moreover, $a = \tau^{-1}(b) \in \tau^{-1}[\tau[A]] = A$. Since $\tau^{-1} \in \mathcal{B}$ we have $\mathbf{s}(a,x) = \mathbf{s}(\tau^{-1}(b),\tau^{-1}(y)) = \tau^{-1}[\mathbf{s}(b,y)] \subset \tau^{-1}[\tau[A]] = A$. Thus $x \in \mathbf{b}(A)$ so $y = \tau(x) \in \tau[\mathbf{b}(A)]$. $\square$

**Theorem.** Suppose $\tau$ is a betweenness preserving permutation of the set of points. The following statements hold:

(i) If $a$ and $b$ are distinct points then $\tau[\mathbf{l}(a,b)] = \mathbf{l}(\tau(a),\tau(b))$; in particular, if $L$ is a line then $\tau[L]$ is a line.

(ii) If $o$ and $a$ are distinct points then $\tau[\mathbf{r}(o,a)] = \mathbf{r}(\tau(o),\tau(a))$. In particular, if $R$ is a ray then $\tau[R]$ is a ray and $\tau(\mathbf{o}(R)) = \mathbf{o}(\tau[R])$.

(iii) If $L$ is a line and $a$ is a point not on $L$. Then $\tau(a)$ is not on $\tau[L]$ and $\tau[\mathbf{h}(L,a)] = \mathbf{h}(\tau[L],\tau(a))$. In particular, if $H$ is a half space then $\tau[H]$ is a half space and $\tau[\mathbf{b}(H)] = \mathbf{b}(\tau[H])$.

(iv) If $<$ is an geometric linear ordering of the line $L$ then

$$\{(\tau(a),\tau(b)) : a,b \in L \text{ and } a < b\}$$

is a geometric ordering of the line $\tau[L]$.

(v) If $F$ is an fundamental set then $\tau[F]$ is an fundamental set; if $S$ is a side of $F$ then $\tau[S]$ is a side of $\tau[F]$; if $v$ is a vertex of $F$ then $\tau(v)$ is a vertex of $\tau[F]$.

**Proof.** Exercise. $\square$

**Definition.** Suppose $\tau \in \mathcal{B}$, $L$ is a line and $\tau[L] = L$. As a consequence of (iv) in the preceding Theorem we see that *either*

$$\tau(a) < \tau(b) \ \Leftrightarrow \ a < b$$

whenever $<$ is a geometric linear ordering of $L$ and $a, b \in L$ in which case we say $\tau$ **is order preserving on** $L$ *or*

$$\tau(b) < \tau(a) \ \Leftrightarrow \ a < b$$

whenever $<$ is a geometric linear ordering of $L$ and $a, b \in L$ in which case we say $\tau$ **is order reversing on** $L$.

Here are three very important consequences of the Continuity Axiom.

**Theorem.** Suppose

(i) $\tau \in \mathcal{B}$, $L$ is a line, $\tau[L] = L$ and $\tau$ is order preserving on $L$;

(ii) $\tau(a) \neq a$ for each $a \in L$; and

(iii) $o \in L$ and $<$ is the geometric linear ordering of $L$ such that $o < \tau(o)$.

Then for each $a \in L$ there is a unique integer $n$ such that

$$\tau^n(o) \leq a < \tau^{n+1}(o).$$

Moreover,

$$a < \tau(a) \quad \text{for } a \in L.$$

**Proof.** Let $R = \{x \in L : o < x\}$. Let $A = \{\tau^n(o) : n \in \mathbf{N}\}$. Since $\tau$ preserves $<$ we find that $A \subset R$.

Suppose $a \in R$. By an earlier Theorem, $a$ is not an upper bound for $A$ so there is a least nonnegative integer $n$ such that $a < \tau^{n+1}(o)$. It follows that $\tau^n(o) \leq a$. Thus $\tau^n(o) \leq a < \tau^{n+1}(o)$. The uniqueness of $n$ follows directly from the fact that $\tau$ preserved $<$ as does the fact that $\tau^{n+1}(o) \leq \tau(a)$ so $a < \tau(a)$.

In case $a = o$ the assertion to be proved is trivial.

In case $a \in R^{\mathbf{o}}$ we apply the results of the first paragraph with $\tau$ replaced by $\tau^{-1}$. $\square$

**Theorem.** Suppose $\tau \in \mathcal{B}$, $L$ is a line, $\tau[L] = L$ and $\tau$ is order reversing on $L$.

Then there is a point $o$ in $L$ such that $\tau(o) = o$.

**Proof.** This follows directly from earlier theory. $\square$

**Theorem.** Suppose $(H, R)$ is a flag, $o$ is the origin of $R$, $\tau \in \mathcal{B}$, $\tau(o) = o$,

$$R \subset \tau[H] \quad \text{and} \quad \tau[R] \subset H.$$

Then there is $S$ such that $S \in \mathbf{R}(o)$, $S \subset \mathbf{a}(R, \tau[R])$ and

$$\tau[S] = S.$$

**Proof.** Let $\prec$ be the geometric linear ordering on $\mathbf{A}(H, R)$ and recall that, by a previous Theorem, $\prec$ is complete. We have $\tau^2[R] \subset \tau[H]$ so $\tau^2[H] \in \mathbf{A}(H, R)$. Thus *either* $\tau^2[R] \subset \mathbf{a}(\tau[R], R)$ *or* $R \subset \mathbf{a}(\tau[R], \tau^2[R])$.

Suppose $(\tau^2[R] \subset \mathbf{a}(\tau[R], R)$ holds. Let $\mathcal{T} = \{S \in \mathbf{A}(H, R) : S \prec R\}$. We have

$$S \in \mathcal{T} \ \Rightarrow \ S \subset \mathbf{a}(\tau[R], R) \ \Rightarrow \tau[S] \in \mathbf{a}(\tau[R], \tau^2[R]) \ \Rightarrow \ \tau[S] \prec \tau^2[R] \ \Rightarrow \ \tau[S] \prec R \ \Rightarrow \ \tau[S] \in \mathcal{T}.$$

Moreover, by an earlier Theorem,

$$S, T \in \mathcal{T} \text{ and } S \prec T \ \Rightarrow S \subset \mathbf{a}(\tau[R], T) \ \Rightarrow T \in \mathbf{a}(R, S) \ \Rightarrow \tau[T] \subset \mathbf{a}(\tau[R], \tau[S]) \ \Rightarrow \tau[T] \prec \tau[S].$$

Thus

$$\{(S,\tau[S]) : S\in\mathcal{T}\}$$

is order reversing function from $\mathcal{T}$ to itself. By earlier theory, there is a member $S$ of $\mathcal{T}$ such that $\tau[S]=S$.

Suppose $R\subset\mathbf{a}(\tau[R],\tau^2[R])$ holds. Let $U=\tau[R]$ and let $I=\tau[H]$ and let $\sigma=\tau^{-1}$. Then

$$U\subset\sigma[I],\quad \sigma[U]\subset I,\quad \sigma^2[U]\subset\mathbf{a}(\sigma[U],U)$$

so we may apply the result just obtained with $\tau,R,H$ replaced by $\sigma,U,I$ to secure $S\in\mathbf{R}(o)$ such that $\sigma[S]=S$ and $S\subset\mathbf{a}(U,\sigma[U])$. It follows that $\tau[S]=S$ and $S\subset\mathbf{a}(R,\tau[R])$.

**Orientation.**

Let $o$ be a point.

**Definition.** We let

$$\mathbf{F}(o)=\{(H,R):(H,R)\text{ is a flag and }o=\mathbf{o}(R)\}.$$

**Definition.** Let

$$\mathbf{K}=\{(1,1),(1,-1),(-1,1),(-1,-1)\}\quad\text{and let}\quad\mathbf{K}^+=\{(1,1),(-1,-1)\}.$$

Note that

$$(ac,bd)\in\mathbf{K}\quad\text{whenever }(a,b),(c,d)\in\mathbf{K}.$$

For each flag $F=(H,R)$ and each $(a,b)\in\mathbf{K}$ we let

$$F^{(a,b)}=\begin{cases}(H,R) & \text{if }(a,b)=(1,1);\\ (H,R^{\mathbf{o}}) & \text{if }(a,b)=(1,-1);\\ (H^{\mathbf{o}},R) & \text{if }(a,b)=(-1,1);\\ (H^{\mathbf{o}},R^{\mathbf{o}}) & \text{if }(a,b)=(-1,-1).\end{cases}$$

Note that

(1) $$F^{(a,b)})^{(c,d)}=F^{(ac,bd)}\quad\text{whenever }(a,b),(c,d)\in\mathbf{K}\}.$$

**Definition.** $G=(I,S)\in\mathbf{F}(o)$. We let

$$\mathbf{p}(G)=\{(H,R)\in\mathbf{F}(o):S\subset H\text{ and }R\subset I^{\mathbf{o}}\}.$$

For each $(a,b)\in\mathbf{K}$ we let

$$\mathbf{P}^{(a,b)}(G)=\{F\in\mathbf{F}(o):F^{(a,b)}\in\mathbf{p}(G)\}.$$

**Proposition.** Suppose $F,G\in\mathbf{F}o$ and $(a,b)\in\mathbf{K}^+$. Then

$$F\in\mathbf{n}(G)\ \Leftrightarrow\ F^{(a,b)}\in\mathbf{n}(G^{(a,b)}).$$

**Proof.** The point here is that if $S\in\mathbf{R}(o)$ and $H$ is a halfplane containing $o$ then $S\subset H$ if and only if $S^{\mathbf{o}}\subset H^{\mathbf{o}}$. $\square$

**Proposition.** Suppose $G\in\mathbf{F}(o)$. Then

$$\{\{G^{(a,b)}\}:(a,b)\in\mathbf{K}\}\cup\{\mathbf{p}^{(a,b)}:(a,b)\in\mathbf{K}\}$$

has eight members and is a partitition of $\mathbf{F}(o)$.

**Theorem.** Suppose $L_1, L_2, L_3$ are three distinct lines; $H_i \in \mathbf{H}(L_i),\ i = 1,2,3$; $L_1 \cap L_2 \cap L_3 \neq \emptyset$;

$$L_1 \cap H_2 \cap H_3 \neq \emptyset \quad \text{and} \quad L_2 \cap H_2 \cap H_3{}^{\mathbf{o}} \neq \emptyset.$$

Then

$$L_3 \cap H_1 \cap H_2 \neq \emptyset.$$

**Proof.** Let $a_1 \in L_1 \cap H_2 \cap H_3$ and let $a_2 \in L_2 \cap H_2 \cap H_3{}^{\mathbf{o}}$. Then $\mathbf{s}(a_1, a_2)$ is a subset of $H_1 \cap H_2$ and meets $L_3$ in a point $a_3$. Since $\mathbf{s}(a_1, a_3) \subset \mathbf{s}(a_1, a_2) \subset H_1$ we infer that $a_3 \in H_1$. since $\mathbf{s}(a_2, a_3) \subset \mathbf{s}(a_1, a_2) \subset H_2$ we infer that $a_3 \in H_2$. $\square$

**Definition.** For each $G \in \mathbf{F}(o)$ we let

$$\mathbf{o}^+(G) = \{G^{(1,1)}\} \cup \{G^{(-1,-1)}\} \cup \mathbf{p}^{(1,1)} \cup \mathbf{p}^{(-1,-1)}$$

and we let

$$\mathbf{o}^-(G) = \{G^{(1,-1)}\} \cup \{G^{(1,-1)}\} \cup \mathbf{p}^{(1,-1)} \cup \mathbf{p}^{(1,-1)}$$

If $F, G \in \mathbf{F}(o)$ we write

$$F \sim_{\mathbf{o}} G$$

and say $F$ **has the same orientation as** $G$ if $F \in \mathbf{o}^+(G)$ and we write

$$F \not\sim_{\mathbf{o}} G$$

and say $F$ **has the opposite orientation to** $G$ if $F \in \mathbf{o}^-(G)$

**Theorem.** Suppose $F, G \in \mathbf{F}(o)$. Then

$$F \sim_{\mathbf{o}} G \;\Leftrightarrow\; F^{\mathbf{o}} \sim_{\mathbf{o}} G^{\mathbf{o}}.$$

**Proof.** Simple exercise for the reader. $\square$

**Theorem.** Then $\sim_{\mathbf{o}}$ is a equivalence relation on $\mathbf{F}(o)$. corresponding to which there are exactly two equivalence classes.

Furthermore, if $G \in \mathbf{F}(o)$ then $\mathbf{o}^+(G)$ and $\mathbf{o}^-(G)$ are the equivalence classes for $\sim_{\mathbf{o}}$.

**Proof.** Suppose $(H_i, R_i),\ i = 1,2,3$, are flags. We need to show that

$$(H_1, R_1) \sim_{\mathbf{o}} (H_1, R_1); \tag{1}$$

$$(H_2, R_2) \sim_{\mathbf{o}} (H_1, R_1) \;\Rightarrow\; (H_2, R_2) \sim_{\mathbf{o}} (H_1, R_1); \tag{2}$$

$$(H_2, R_2) \sim_{\mathbf{o}} (H_1, R_1) \text{ and } (H_3, R_3) \sim_{\mathbf{o}} (H_2, R_2) \;\Rightarrow\; (H_3, R_3) \sim_{\mathbf{o}} (H_1, R_1). \tag{3}$$

That (1) and (2) hold is obvious.

Let us prove (3). Suppose $(H_1, R_1) \in \mathbf{o}^+((H_2, R_2))$ and $(H_2, R_2) \in \mathbf{o}^+((H_3, R_3))$ but $(H_1, R_1) \in \mathbf{o}^-((H_3, R_3))$. Then

$$R_1 \subset H_2 \cap H_3; \quad R_2 \subset H_1{}^{\mathbf{o}} \cap H_3; \quad R_3 \subset H_2{}^{\mathbf{o}} \cap H_1.$$

Then $L_1 \cap H_3 \cap H_2 \neq \emptyset$ and $L_3 \cap H_1 \cap H_2{}^{\mathbf{o}} \neq \emptyset$ so the Lemma gives $L_2 \cap H_3 \cap H_1 \neq \emptyset$ which is incompatible with $R_2 \subset H_1{}^{\mathbf{o}} \cap H_3$.

**Definition.** Suppose $\tau \in \mathcal{B}(o)$. We say a $\tau$ is **orientation preserving** if $(\tau[H], \tau[R]) \sim_{\mathbf{o}} (H, R)$ for any $(H, R) \in \mathbf{F}(o)$. We say a $\tau$ is **orientation reversing** if $(\tau[H], \tau[R]) \not\sim_{\mathbf{o}} (H, R)$ for any $(H, R) \in \mathbf{F}(o)$.

**Theorem.** Any member of $\mathcal{B}(o)$ is either orientation preserving or orientation reversing. Furthermore, if $\sigma, \tau \in \mathcal{B}(o)$ then $\tau \circ \sigma$ is orientation preserving if and only if $\sigma$ and $\tau$ are both orientation preserving or orientation reversing.