**Angle sums and more.**

Among other things, we will prove the following Theorems.

**Theorem One.** If the angle sum of some triangle is $\pi$ then the angle sum of some right triangle is $\pi$.

**Theorem Two.** If the angle sum of some triangle is $\pi$ then there is a rectangle.

**Theorem Three.** If there is a rectangle the the angle sum for every right triangle is $\pi$.

**Theorem Four.** If the angle sum for every right triangle is $\pi$ then the angle sum for any triangle is $\pi$.

**Theorem Five.** If the angle sum for every right triangle is $\pi$ then the parallel postulate holds.

**Theorem Six.** If the parallel postulate holds then the angle sum for every triangle is $\pi$.

The proofs follow. Along the way we will prove some Theorems which are interesting in their own right.

**Definition.** Suppose $T$ is a triangle. We let

$$\delta(T)$$

be $\pi$ minus the sum of the angles of $T$. Don't ever forget that, by the Saccheri Legendre Theorem, $\delta(T) \geq 0$. For some reason that escapes me the book calls $\delta(T)$ the **defect of** $T$. Does that mean there might be something wrong with $T$ if its "defect" $\delta(T)$ is positive?

**Theorem. The "additivity of the defect" in the book.** Suppose $T$ is a triangle with vertices $a, b, c$ and $d \in \mathbf{s}(b, c)$. Let $T'$ and $T''$ be the triangles with vertices $a, b, d$ and $a, c, d$, respectively. Then

$$\delta(T) = \delta(T') + \delta(T'').$$

**Proof.** Let $A'$ and $A''$ be the angles of $T'$ and $T''$, respectively, corresponding to the vertex $a$. Let $D'$ and $D''$ be the angles of $T'$ and $T''$, respectively, corresponding to the vertex $d$. By the Crossbar Theorem, $|A| = |A'| + |A''|$. Thus

$$\begin{aligned}
\delta(T) &= \pi - (|A| + |B| + |C|) \\
&= \pi - (|A'| + |A''| + |B| + |C|) \\
&= \pi - (|A'| + |A''| + |B| + |C|) + (\pi - (|D'| + |D''|) \\
&= (\pi - (|A'| + |B| + |D'|)) + (\pi - (|A'| + |C| + |D''|)) \\
&= \delta(T') + \delta(T'').
\end{aligned}$$

□

**Theorem.** Suppose $T$ is a triangle, $a$ is a vertex of $T$, $A$ is the angle corresponding to $a$ and $d$ is the perpendicular dropped from $a$ to the line containing the opposite side.

If $A$ is not acute then $d$ is in the opposite side.

If $d$ is not in the opposite side and $d$ is not a vertex of $T$ then $A$ is acute and there is exactly one angle $B$ of $T$ such that $B$ is obtuse.

**Proof.** If $d$ is a vertex of $T$ then $A$ is acute because the angle of $T$ corresponding to $d$ is a right angle and the angle sum of $T$ does not exceed $\pi$.

Suppose $d$ is not in the side of $T$ opposite $a$ and $d$ is not a vertex of $T$. Then there are $b, c$ such that $a, b, c$ are the vertices of $T$ and $b \in \mathbf{s}(c, d)$. Let $B$ be the angle corresponding to $b$. Since the triangle with vertices $a, d, b$ has a right angle at $d$ and since its angle sum does not exceed $\pi$ we infer that $\mathbf{a}(d, b, a)$ is acute. Thus $B$ is obtuse. Finally, by the exterior angle theorem, $A$ is acute. □

**Remark.** We needed this for the Pythagorean Theorem. See how easy it is for a proof in this subject to have a gap?

**Proof of Theorem One.** Let $T$ be a triangle with angle sum $\pi$. Then, by the preceding Theorem, there is a vertex $a$ of $T$ such that the perpendicular dropped from $a$ to the line containing the opposite side is in the opposite side. By the Theorem on the "additivity of the defect" we obtain two right triangles with defect zero. $\square$

**Proof of Theorem Two.** Let $T$ be a right triangle with zero defect. Let $R$ be the union of $T$ with one of its sides and the reflection across the line containing that side. Make sure you check that you get a quadrilateral! $\square$

**Theorem.** Suppose $Q$ is a quadrilateral; $a, b, c, d$ are the distinct vertices of $Q$;

$$\mathbf{s}(a,b), \quad \mathbf{s}(c,d), \quad \mathbf{s}(a,c), \quad \mathbf{s}(b,c)$$

are the sides of $Q$; $A, B, C, D$ are the angles of $Q$ corresponding to $a, b, c, d$; and

$$|A| = |B|.$$

Then

$$|C| = |D| \;\Leftrightarrow\; |\mathbf{s}(a,c)| = |\mathbf{s}(b,d)|;$$
$$|C| < |D| \;\Leftrightarrow\; |\mathbf{s}(a,c)| > |\mathbf{s}(b,d)|;$$
$$|D| < |C| \;\Leftrightarrow\; |\mathbf{s}(a,c)| < |\mathbf{s}(b,d)|.$$

**Proof.** Let $\rho$ be reflection across the perpendicular bisector of $\mathbf{s}(a,b)$. Note that $\rho$ interchanges $R = \mathbf{r}(a,c)$ and $S = \mathbf{r}(b,d)$. (Why?)

If $|\mathbf{s}(a,c)| = |\mathbf{s}(b,d)|$ then $\rho(c) = d$ and $\rho(d) = c$. (Why?) Thus $\rho$ interchanges $C$ and $D$ so $|C| = |D|$. Thus

(1) $$|\mathbf{s}(a,c)| = |\mathbf{s}(b,d)| \;\Rightarrow\; |C| = |D|.$$

If $|\mathbf{s}(a,c)| < |\mathbf{s}(b,d)|$ then $|\mathbf{s}(b,\rho(c))| = |\rho[\mathbf{s}(a,c)]| < |\mathbf{s}(b,d)|$ so, as $\rho(c) \in S$, $d \in \mathbf{s}(b,\rho(c))$. From our earlier theory on polygons we know that $\mathbf{s}(c,\rho(c)) \subset Q$. From the exterior angle theorem we infer that

$$|C| > |\mathbf{a}(a,c,\rho(c))| = |\rho[\mathbf{a}(a,c,\rho(c))]| = |\mathbf{a}(b,\rho(c),c)]| > |\mathbf{a}(c,d,\rho(c))| = |D|.$$

Thus

(2) $$|\mathbf{s}(a,c)| < |\mathbf{s}(b,d)| \;\Rightarrow\; |D| < |C|.$$

Permuting $a, b, c, d$ appropriately we infer from (2) thatthat

(3) $$|\mathbf{s}(b,d)| < |\mathbf{s}(a,c)| \;\Rightarrow\; |C| < |D|.$$

Suppose $|C| = |D|$. By the results of preceding two paragraphs we infer that neither $|\mathbf{s}(a,c)| < |\mathbf{s}(c,d)|$ nor $|\mathbf{s}(c,d)| < |\mathbf{s}(a,c)|$ holds. By trichotomy of segment ordering we conclude that $|\mathbf{s}(a,c)| = |\mathbf{s}(b,d)|$. Thus

(4) $$|C| = |D| \;\Rightarrow\; |\mathbf{s}(a,c)| = |\mathbf{s}(b,d)|.$$

Suppose $|C| < |D|$. By (1) $\mathbf{s}(a,b)$ is not congruent to $\mathbf{s}(c,d)$. By trichotomy of segment ordering either $|\mathbf{s}(a,c)| < |\mathbf{s}(b,d)|$ or $|\mathbf{s}(b,d)| < |\mathbf{s}(a,c)|$. The first of these is excluded by (2). Thus

(5) $$|C| < |D| \;\Rightarrow\; |\mathbf{s}(b,c)| < |\mathbf{s}(a,c)|$$

Permuting $a, b, c, d$ appropriately we infer from (5) thatthat

(5) $$|D| < |C| \;\Rightarrow\; \mathbf{s}(a,d) < \mathbf{s}(b,c).$$

□

**Corollary.** Opposite sides of a rectange are congruent.
**Proof.** One can take all of $A, B, C, D$ to be right angles above. □

**Proof of Theorem Three.** Let $R$ be a rectangle with vertices $a, b, c, d$ such that $\mathbf{s}(a,b) \simeq \mathbf{s}(c,d)$ and $\mathbf{s}(a,c) \simeq \mathbf{s}(b,d)$.

Let $T$ be a triangle with vertices $e, f, g$ and with the angle corresponding to $e$ being a right angle. Choose positive integers $m$ and $n$ such that $m|\mathbf{s}(a,b)| \geq < \mathbf{s}(e,f) >$ and $n|\mathbf{s}(a,c)| \geq |\mathbf{s}(e,d)|$.

Let $\tau_{a,b}$ be translation from $a$ to $b$ and let $\tau_{a,c}$ be translation from $a$ to $c$.

$$a' = a. \quad b' = \tau_{a,b}^m(a); \quad c' = \tau_{a,c}^n(c); \quad d' = \tau_{a,c}^n(\tau_{a,b}^m(a)).$$

We leave it to the reader to use the above Corollary and our theory developed to this point to show that $a', b', c', d'$ are the vertices of a rectangle $R'$.

Let $e' = a'$; let $d' \in \mathbf{s}(a', b')$ be such that $\mathbf{s}(e', d') \simeq \mathbf{s}(e, d)$; and let $f' \in \mathbf{s}(a', c')$ be such that $\mathbf{s}(e', f') \simeq \mathbf{s}(e, f)$. Let $T'$ be the triangle with vertices $d', e', f'$. Evidently, $T \simeq T'$.

Since $\mathbf{s}(c', b')$ is a diagonal of $R'$, the triangle with vertices $e', b', c'$ has zero defect. By the additivity of the defect so does the triangle with vertices $e', c', c'$. By the additivity of the defect so does $T'$. □

**Proof of Theorem Four.** Just drop an appropriate perpendicular and use the additivity of the defect.

**Aristotle's Axiom.** Suppose $R, S$ are the sides of an acute angle and $l$ is a positive real number. There is a point $p$ on $R$ such that if $q$ is the foot of the perpendicular dropped from $p$ to the line containing $S$ then $|\mathbf{s}(p,q)| > l$.
**Proof.** Let $o$ be the vertex of $\mathbf{a}(R,S)$. Let $e \in R$, let $b$ be the foot of the perpendicular dropped from $e$ to the line containing $S$ and note that $b \in S$. (Why?) Let $\alpha_a$ be the half turn about $e$. Let $c = \alpha_a(o)$, let $d$ be the foot of the perpendicular dropped from $c$ to the line containing $S$ and let $a = \alpha_e(b)$. Note that $a, b, c, d$ are the vertices of a quadrilateral. Let $A, B, C, D$ be the corresponding angles. Note that $A, B, D$ are right angles. Thus $C$ has measure at most $\pi/2$. By the previous Theorem, $2|\mathbf{s}(e,b)| = |\mathbf{s}(a,b)| \leq |\mathbf{s}(c,d)|$.

By iterating this construction and appealing the Archimedean Property of segment ordering we obtain the desired point $p$ on $R$. □
**Remark.** Note that, again by the previous Theorem, $|\mathbf{s}(b,d)| \leq |\mathbf{s}(a,c)| = |\mathbf{s}(o,b)|$.

We're now ready to prove Theorem Five.
**Proof of Theorem Five.** Suppose $L$ is a line and $p \notin L$. Let $o \in L$ be the foot of the perpendicular dropped from $p$ to $L$. Let $M$ be the line through $p$ making a right angle with $\mathbf{l}(o,p)$. Note that $M$ is parallel to $L$ by the alternating interior angle theorem. Suppose $O$ were another parallel to $L$ through $p$. Then $O \subset H = \mathbf{h}(L,p)$. Let $I = \mathbf{h}(M,o)$. Let $R = I \cap O$ and note that $R$ is a ray with origin $p$ and $R \subset H \cap I$. Use Aristotle's Axiom to obtain points $r \in O$ and $s \in M$ such that $s$ is the foot of the perpendicular dropped from $r$ to $s$ and $|\mathbf{s}(r,s)| > |\mathbf{s}(o,p)|$. Let $q$ be the foot of the perpendicular dropped from $r$ to $\mathbf{l}(o,p)$ and note that $r \in \mathbf{s}(o,p)$. Note that $p, q, r, s$ are the vertices of a quadrilateral $Q$. Since $r, s, p$ and $r, q, p$ are the vertices of right triangles we find that $Q$ is a rectangle. Since $\mathbf{s}(p,q)$ and $\mathbf{s}(r,s)$ are opposite sides of this rectangle we infer from a Theorem above that $|\mathbf{s}(p,q)| = |\mathbf{s}(r,s)|$ which is incompatible with $|\mathbf{s}(p,q)| < |\mathbf{s}(o,p)| < |\{Sr, s|$.
□

We have already proved Theorem Six.