Test Two Answer Key Mathematics 114.01 Spring 2005

TO GET FULL CREDIT YOU MUST SHOW ALL WORK!

The average was 45.8. The standard deviation was 16.71.

1. 5 points. Let $f(z) = \sin z/z$ for $z \in \mathbf{C} \sim \{0\}$. Is 0 a removable singularity?

Solution. We have

$$\sin z/z = \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right) = 1 - \frac{z^2}{3!} + \frac{z^2}{5!} - \cdots$$

so 0 is a removable singularity.

2. Let

$$f(z) = \frac{z}{(z-1)(z-2)^2}, \quad z \in \mathbf{C} \sim \{0\}.$$

(a)**(15 points.)** Identify the poles of f and their orders and calculate the residues.
(b) **(5 points.)** Calculate $\int_C f(z)\,dz$ where C is the circle of radius 3 centered at the origin which winds clockwise about the origin.

We have

$$f(z) = \frac{1}{z-1} f_1(z) \quad \text{where} \quad f_1(z) = \frac{z}{(z-2)^2}$$

and $f_1(1) = 1 \neq 0$ so 1 is a pole of order 1 and the residue is $f_1(1) = 1$. We have

$$f(z) = \frac{1}{(z-2)^2} f_2(z) \quad \text{where} \quad f_1(z) = \frac{z}{z-1}$$

and $f_2(2) = 2 \neq 0$ so 2 is a pole of order 2 and the residue is $f_2'(2) = -1$.

The integral is

$$-2\pi i(\mathrm{Res}(f,1) + \mathrm{Res}(f,2)) = -2\pi i(1-1) = 0;$$

the leading minus is because the curve is going clockwise.

3. 10 pts. Let $f(z) = e^{1/z}$ for $z \in \mathbf{C} \sim \{0\}$. There exist a coefficient sequence $c_0, c_1, c_2, \ldots$ and positive radii R such that

$$f(z) = \sum_{n=0}^{\infty} c_n (z-1)^n, \quad |z-1| < R.$$

Identify all such R and calculate c_0, c_1, c_2.

Solution. f is analytic in a disc of radius 1 centered at 1 and in no larger disc so $R = 1$. Moreover,

$$c_0 = f(1) = e; \quad c_1 = f'(1) = -e; \quad c_2 = \frac{f''(1)}{2!} = \frac{3e}{2}.$$

4. 10 points. Suppose a, b, c are real numbers such that $a \neq 0$ and $b^2 - 4ac < 0$. Calculate

$$\int_{-\infty}^{\infty} \frac{dx}{ax^2 + bx + c}.$$

Solution. Let $f(z) = \frac{1}{az^2+bz+c}$. Let $D = 4ac - b^2 > 0$, let

$$r_+ = \frac{-b + i\sqrt{D}}{2a} \quad \text{and let} \quad r_- = \frac{-b - i\sqrt{D}}{2a}.$$

Then

$$f(z) = \frac{1}{a(z - r_+)(z - r_-)}$$

so f has simple poles at r_+ and r_- with respective residues

$$\frac{1}{a(r_+ - r_-)} = \frac{1}{i\sqrt{D}} \quad \text{and} \quad \frac{1}{a(r_- - r_+)} = -\frac{1}{i\sqrt{D}}.$$

Suppose $R > |r_+|$. Let I_R be the segment from $(-R, 0)$ to $(R, 0)$ and let S_R be the semicircle of radius R centered at the origin which goes from $(R, 0)$ to $(-R, 0)$ in the upper half plane. By the residue theorem,

$$\int_{I_R+S_R} f(z)\,dz = 2\pi i \frac{1}{i\sqrt{D}} = \frac{2\pi}{\sqrt{D}}.$$

Since $\lim_{|z|\to\infty} |z||f(z)| = 0$ we find that

$$\lim_{R\to\infty} \int_{S_R} f(z)\,dz = 0.$$

Thus

$$\int_{-\infty}^{\infty} \frac{dx}{ax^2 + bx + c} = \lim_{R\to\infty} \int_{I_R} f(z)\,dz = \lim_{R\to\infty} \int_{I_R+S_R} f(z)\,dz = \frac{2\pi}{\sqrt{D}}.$$

5. Let f be the 2π periodic function such that

$$f(x) = \begin{cases} 0 & \text{if } -\pi \le x < 0, \\ x & \text{if } 0 \le x < \pi/2, \\ 0 & \text{if } \pi/2 \le x < \pi. \end{cases}$$

(a) **(10 points)** Compute (f, E_n), n any integer. (I suggest you use the jump formula.)
(b) **(10 points)** Evaluate

$$\frac{1}{2\pi} \sum_{n=-\infty}^{\infty} (f, E_n) i^n.$$

Solution.

$$(f, E_0) = \int_{-\pi}^{\pi} f(x)\,dx = \frac{x^2}{2}\bigg|_{x=0}^{x=\pi/2} = \frac{\pi^2}{8}.$$

For $n \neq 0$ we have, as f has a jump of $-\pi/2$ at $x = \pi/2$

$$(f, E_n) = \frac{1}{in}\left((f', E_n) + (-\pi/2)e^{-in\pi/2}\right) = \frac{1}{in}\left((f', E_n) - (\frac{\pi}{2})(-i)^n\right).$$

As f' had a jump of 1 at $x = 0$ and -1 at $x = \pi/2$ we have

$$(f', E_n) = \frac{1}{i\pi n}\left((f'', E_n) + e^{-in0} - e^{-in\pi/2}\right) = \frac{1}{i\pi n}\left(1 - (-i)^n\right).$$

Thus if $n \neq 0$ we have

$$(f, E_n) = \frac{1}{in}\left(\frac{1}{in}(1 - (-i)^n) - (\frac{\pi}{2})(-i)^n\right).$$

By the Fourier inversion formula the answer to (b) is the average value of f at $\pi/2$ since $i^n = E_n(\pi/2)$. This average value is $\pi/4$.

6. 10 points. Let $f(x) = 1 - x$ for $0 < x < 1$ and let

$$a_n = 2\int_0^1 f(x)\cos n\pi x\, dx, \quad n = 1, 2, 3, \ldots.$$

Determine

$$A(x) = \sum_{n=1}^{\infty} a_n \cos n\pi x \quad \text{and} \quad B(x) = \sum_{n=1}^{\infty} n\pi a_n \sin n\pi x$$

for $x \in (0, 1)$.

Solution. Suppose $0 < x < 1$. Then $f(x) = \frac{a_0}{2} + A(x)$. Since $a_0 = 2\int_0^1 f(x)\, dx = 1$ we find that $A(x) = \frac{1}{2} - x$. Note that $B(x)$ is minus term-by-term differentiation $A(x)$; thus $B(x) = 1$.

7. Suppose $a > 0$ and $b \in \mathbf{R}$. Let $f(t) = e^{-a|t|}\sin bt$ for $t \in \mathbf{R}$.

(a) **(10 points.)** Calculate $\hat{f}$. (I suggest you begin by expressing f in terms of complex exponentials.)

(b) **(10 points.)** Evaluate

$$\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega}\, d\omega.$$

Solution. We have

$$\begin{aligned}
2i\hat{f}(\omega) &= 2i\int_{-\infty}^{\infty} e^{-a|t|}\sin bt\, e^{-i\omega t}\, dt \\
&= \int_{-\infty}^{\infty} e^{-a|t|}(e^{ibt} - e^{-ibt})e^{-i\omega t}\, dt \\
&= \int_{-\infty}^{0} e^{at+i(b-\omega)t}\, dt + \int_0^{\infty} e^{-at+i(b-\omega)t}\, dt \int_{-\infty}^{0} e^{at+i(-b-\omega)t}\, dt + \int_0^{\infty} e^{-at+i(-b-\omega)t}\, dt \\
&= \frac{e^{at+i(b-\omega)t}}{a+i(b-\omega)}\Big|_{-\infty}^{0} + \frac{e^{-at+i(b-\omega)t}}{-a+i(b-\omega)}\Big|_0^{\infty} + \frac{e^{at+i(-b-\omega)t}}{a+i(-b-\omega)}\Big|_{-\infty}^{0} + \frac{e^{-at+i(-b-\omega)t}}{-a+i(-b-\omega)}\Big|_0^{\infty} \\
&= \frac{1}{a+i(b-\omega)} - \frac{1}{-a+i(b-\omega)} + \frac{1}{a+i(-b-\omega)} - \frac{1}{-a+i(-b-\omega)} \\
&= \frac{2a}{a^2+(b-\omega)^2} + \frac{2a}{a^2+(b+\omega)^2};
\end{aligned}$$

By the Fourier inversion formula,

$$\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega}\, d\omega = 2\pi f(1) = 2\pi e^{-a}\sin b.$$