Additional Homework Problems (AHP)

1. For each of the following equations, find a function for which this equation represents a level set of that function.

   1. \( x^2 y - y^3 = xe^{xy} \)
   2. \( x^2 = y^2 - 1 \)
   3. \( x_1 x_2 - x_3 x_4 = x_5 \)

2. For each of the following functions, find another function whose zero level set is the same as the graph of the given function.

   1. \( f_1(x, y) = x^2 + y^2 \)
   2. \( f_2(x) = x^2 - x^3 \)
   3. \( f_3(x, y, z) = x + y - z \)

3. For each of the following, use the given information to find the matrix that corresponds to the indicated linear transformation.

   1. \( f_1 : \mathbb{R}^2 \to \mathbb{R}^3, f_1(1, 0) = (2, 3, 4), f_1(0, 1) = (5, 2, 3) \)
   2. \( f_2 : \mathbb{R}^2 \to \mathbb{R}^1, f_2(1, 1) = 3, f_2(0, 1) = 5 \)
   3. \( f_3 : \mathbb{R}^2 \to \mathbb{R}^2, f_3(2, 3) = (2, 3), f_3(3, 1) = (5, 3) \)

4. For each of the following you are given a set \( U \) and a point \( \vec{x} \). Determine if the point \( \vec{x} \) is in \( U \) or not in \( U \). Also determine if it is on the boundary.

   1. \( U = \{(x, y)|x^2 + y^2 \leq 1\} \) and \( \vec{x} = (0.5, 0.5) \)
   2. \( U = \{(x, y)|x^2 + y^2 < 1\} \) and \( \vec{x} = (\sqrt{0.5}, \sqrt{0.5}) \)
   3. \( U = \{(x, y)|x^2 + y^2 \geq 0\} \) and \( \vec{x} = (0, 0) \)
   4. \( U = \{(x, y)|x^2 + y^2 \leq 1 \text{ and } x > 0\} \) and \( \vec{x} = (0, 0.3) \)

5. For each of the sets in Problem 4 above, determine if the set is open, closed, or neither.

6. Suppose we are trying to maximize the \( C^1 \) function \( f : \mathbb{R}^2 \to \mathbb{R}^1 \) on the solid closed unit square \( S = [0, 1] \times [0, 1] \). Suppose we know that \( \nabla f(1/2, 1/2) = (-2, 4) \). Given only this information, can we eliminate the possibility that \( (1/2, 1/2) \) might be a max? Explain your reasoning.

7. Suppose we are trying to maximize the \( C^1 \) function \( f : \mathbb{R}^2 \to \mathbb{R}^1 \) on the solid closed unit square \( S = [0, 1] \times [0, 1] \). Suppose we know that \( \nabla f(1/3, 1/4) = (0, 0) \). Given only this information, can we eliminate the possibility that \( (1/3, 1/4) \) might be a max? Explain your reasoning.
8. Suppose we are trying to maximize the $C^1$ function $f : \mathbb{R}^2 \to \mathbb{R}$ on the solid closed unit square $S = [0, 1] \times [0, 1]$. Suppose we know that $\nabla f(1, 1) = (-2, 2)$. Given only this information, can we eliminate the possibility that $(1, 1)$ might be a max? Explain your reasoning. (Hint: What is the directional derivative of $f$ in the direction of the vector $(-2, -1)$, and does moving in that direction keep you within the set?)

9. Suppose we are trying to maximize the $C^1$ function $f : \mathbb{R}^2 \to \mathbb{R}$ on the solid closed unit square $S = [0, 1] \times [0, 1]$. Suppose we know that $\nabla f(1, 1) = (3, 1)$. Given only this information, can we eliminate (only using techniques from this section) the possibility that $(1, 1)$ might be a max? Explain your reasoning. (Hint: Does there exist a direction $\vec{v}$ from this point which keeps you in the set, and in which the directional derivative is positive?)

10. Find the absolute maximum value of the function $f(x, y) = xy - 4x^2 - y^2$ on the $xy$-plane.

11. For which of the following optimization problems (maximizing $f$ on the domain $U$) can we conclude immediately that the absolute maximum must exist? If so, explain your reasoning; if not, explain what necessary property is not satisfied.

   1. $f(x, y) = x^3 - y^8 + x^2y$, $U = [1, 3] \times [2, 3]$
   2. $f(x, y) = 1/(x^2 + y^2)$, $U = \{(x, y)| (x-1)^2 + y^2 < 1\}$
   3. $f(x, y) = xy$, $U = \{(x, y)| x \leq 1\}$
   4. $f(x, y) = 1/(x^2 + y^2)$, $U = \{(x, y)| (x-1)^2 + (y-1)^2 \leq 1\}$

12. For the following system, find all allowable sets of endogenous variables.

   \[\begin{align*}
   2x - 3y + 4z &= b_1 \\
   3x + 4y - z &= b_2
   \end{align*}\]

13. For the following system, find all allowable sets of endogenous variables.

   \[\begin{align*}
   w - 5x - 1y + 2z &= b_1 \\
   3w + 2x - y - 2z &= b_2
   \end{align*}\]

14. For the following system, find all allowable sets of endogenous variables.

   \[\begin{align*}
   2w - 2x + y - 3z &= b_1 \\
   w + x - 3y - 4z &= b_2 \\
   2w + 2x - 4y - z &= b_3
   \end{align*}\]