1. (54 pts) The differential equation \( \frac{d^2y}{dx^2} + 8 \frac{dy}{dx} + 16y = -\alpha^2y \) has the general solution

\[
y_{\text{general}}(x) = \begin{cases} 
    c_1 e^{-4x} + c_2 xe^{-4x} & \text{if } \alpha = 0 \\
    c_1 e^{-4x} \cos(\alpha x) + c_2 e^{-4x} \sin(\alpha x) & \text{if } \alpha > 0
\end{cases}
\]

Let \( Lu \equiv u'' + 8u' + 16u \) and consider the eigenvalue problem on \( 0 \leq x \leq 1/4 \):

\[ L\phi = -\lambda \phi \quad \phi(0) = 0 \quad \left. \frac{d\phi}{dx} \right|_{x=1/4} = 0 \]

(a) (3 pts) Find the eigenfunction for \( \lambda_0 = 0 \).

(b) (5 pts) For \( \lambda_k > 0 \) simplify down the general solution to give the eigenfunctions for \( k = 1, 2, \cdots \) and write the equation that determines the eigenvalues.

(Simplify the equation as much as possible.

(Do NOT attempt to solve for the values of the \( \lambda_k \)'s!)

(c) (9 pts) Determine the adjoint operator and adjoint boundary conditions with respect to the standard \( L^2 \) inner product on \( 0 \leq x \leq 1/4 \).

(d) (7 pts) Find the adjoint eigenfunction for \( \lambda_0 = 0 \)

(e) (14 pts) For a given function \( f(x) \) on \( 0 \leq x \leq 1/4 \) consider the expansion \( f(x) = \sum_{k=0}^{\infty} c_k \phi_k(x) \).

Write the formulas for the coefficients \( c_0 \) and \( c_{k \geq 1} \) in terms of integrals.

Write one sentence to explain if you are using bi-orthogonality or self-orthogonality.

Simplify/evaluate the integrals as much as possible.

(f) (16 pts) For the problem

\[ u'' + 8u' + 16u = 256x \quad u(0) = A \quad u'(1/4) = 8 \]

Determine the value(s) of \( A \) needed for a solution to exist. Write one sentence to explain the existence or uniqueness of solutions for this problem.

2. (46 pts) Consider the linear operator: \( Lu \equiv 2u(x) + \int_0^\infty [32xe^{-3t} + 8x^2 te^{-x-t}] u(t) \, dt \).

Note that \( \int_0^\infty y^n e^{-ay} \, dy = \frac{n!}{a^{n+1}} \) where \( n! = 1 \cdot 2 \cdots (n-1) \cdot n \) and \( 0! = 1 \)

(a) (16 pts) Determine all of the eigenvalues and determine the eigenfunctions for the eigenvalues of finite multiplicity in \( L\phi = \lambda \phi \).

(b) (14 pts) Write the adjoint operator. Determine the adjoint eigenfunction for the eigenvalues of finite multiplicity.

(c) (16 pts) Determine the constants \( a, b \) so that

\[ \phi(x) = 17 + ax + be^{-x} \]

is an eigenfunction.