1. (30 pts) For $\alpha > 0$, the general solution of the ODE
\[
\frac{d}{dx} \left( x^2 \frac{dy}{dx} \right) + \alpha^2 x^2 y = 0 \quad \text{is} \quad y = C_1 \frac{\sin(\alpha x)}{x} + C_2 \frac{\cos(\alpha x)}{x}.
\]
(a) (10 pts) Find the eigenvalues and eigenfunctions on $1 \leq x \leq 2$ for
\[
\frac{d}{dx} \left( x^2 \frac{d\phi}{dx} \right) = -\lambda x^2 \phi, \quad \phi(1) = 0, \quad \phi(2) = 0.
\]
(Do not attempt to shift the phase ($x \to x-x_c$), the soln is not translation invariant. Use trig. identities)
(b) (20 pts) Determine the coefficients in the eigenfunction expansion $u(x) = \sum_k c_k \phi_k(x)$ using the eigenfunctions from (a) for the solution $u(x)$ of
\[
\frac{d}{dx} \left( x^2 \frac{du}{dx} \right) = \pi x^2, \quad u(1) = 3, \quad u(2) = -5.
\]
Simplify your answer as much as possible.

2. (37 pts) Consider the linear operator:
\[
Lu \equiv -2u(x) + \int_0^\infty \left[ 10e^{-2x} \cos t + 6e^{-x} \right] u(t) \, dt.
\]
(a) (16 pts) Determine all of the eigenvalues and determine the eigenfunctions for the eigenvalues of finite multiplicity in $L\phi = \lambda \phi$.
(b) (9 pts) Write the adjoint operator. Determine the adjoint eigenfunction for the eigenvalues of finite multiplicity.
(c) (12 pts) Solve for $u(x)$:
\[
Lu = 14e^{-3x}
\]
(Hint: Don’t forget $-2u$)

Note that $\int_0^\infty e^{-at} \cos(bt) \, dt = \frac{a}{a^2 + b^2}$ for $a > 0, \ b \geq 0$

3. (33 pts) For $\alpha > 0$, the general solution of the ODE
\[
\frac{d^2 y}{dx^2} - \frac{2}{x} \frac{dy}{dx} + \alpha^2 y = 0
\]
is $y = C_1 \left[ \sin(\alpha x) - \alpha x \cos(\alpha x) \right] + C_2 \left[ \cos(\alpha x) + \alpha x \sin(\alpha x) \right]$.
Consider the linear operator $L\phi \equiv \phi'' - (2/x)\phi'$ and the eigenvalue problem for $\phi(x)$ on $0 \leq x \leq \pi$:
\[
L\phi = -\lambda \phi, \quad \phi(0) = 0, \quad \frac{d\phi}{dx} \big|_{x=\pi} = 0.
\]
(a) (8 pts) Determine the eigenvalues and eigenfunctions.
(b) (10 pts) Determine the adjoint operator and adjoint boundary conditions with respect to the standard $L^2$ inner product on $0 \leq x \leq \pi$.
(c) (7 pts) Determine $p(x), q(x), \sigma(x)$ that put the problem into Sturm-Liouville form, $(p(x)\phi')' + q(x)\phi = -\lambda \sigma(x)\phi$. Also, give the formula for the adjoint eigenfunction $\psi$ (satisfying $L^\ast \psi = -\lambda \psi$) in terms of $\phi$.
(d) (8 pts) Show how to apply the Fredholm alternative theorem to determine the value of the constant $A$ for which a solution exists for
\[
\frac{d^2 u}{dx^2} - \frac{2}{x} \frac{du}{dx} + 4u = A, \quad u(0) = \pi, \quad \frac{du}{dx} \big|_{x=\pi} = 0.
\]
(Hints: L’Hopital’s rule for the boundary term, $(f/g)'$ in the integral.)