Complex Contour Integrals

-1. Test 3 (Monday, Dec 1) will make use of complex variables and functions (HW 9), and mainly cover complex integrals (this problem set). (This is the final homework.)

0. Reading: See “Methods of contour integration” and related pages on Wikipedia, MathWorld, ...

1. (Residues and Heaviside’s short-cut for partial fractions) First, use Cauchy’s integral formula to evaluate the integral on curve $C$ going around the three poles in terms of their corresponding residues:

$$\frac{1}{2\pi i} \oint_C \frac{dz}{(z-z_a)(z-z_b)(z-z_c)} = A + B + C.$$ 

What are the formulas for $A, B, C$? Show that these are the same as the the partial fraction coefficients that you’d get from doing algebra to match coefficients in the LHS vs. RHS below:

$$\frac{1}{(z-z_a)(z-z_b)(z-z_c)} = \frac{A}{z-z_a} + \frac{B}{z-z_b} + \frac{C}{z-z_c}. $$

Namely: combine the RHS fractions over a common denominator and match the coefficients RHS numerator polynomial to the LHS numerator polynomial, $0z^2 + 0z + 1$, to obtain equations for the coefficients $A, B, C$.

(This short-cut via the residue theorem is called “Heaviside’s cover-up method,” [i.e. cover the pole factor.])

2. Evaluate the integral on each of the following curves from $z = 0$ to $z = 1 + i$:

$$\int_C |z|^2 dz$$

(a) On the diagonal line, $y = x$.

(b) Clockwise on the circle $|z - 1| = 1$.

(c) Counterclockwise on the circle $|z - i| = 1$.

(d) “Polygonal path”: line segments along the real axis then parallel to the imaginary axis.

3. Evaluate on the CCW circle $C: |z| = \sqrt{2}$

$$\oint_C \frac{\sin(\pi z)}{z^2(z-1)(z-\frac{1}{2})} dz$$

4. (“Loop-the-loop”2) Evaluate the integral for the CCW closed contour $r = 2 - \sin^2(\theta/4)$,

$$\oint_C \frac{dz}{z}$$

Hint: Plot the whole closed curve, then deform it.

5. Evaluate the integral

$$\int_0^\pi \frac{d\theta}{7 + \cos \theta}$$

Hint: First use $\sin^2 x = (1 - \cos(2x))/2$.

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1 Yes, the same guy as the step-function.
2 Describes the shape of the curve...
3 Tom likes to make pizza-related references...
10. (Optional, EXTRA CREDIT) Using contour integration to show how to evaluate the integral

\[ \int_0^1 \frac{\ln(x)}{x-1} \, dx. \]

Hint: Use \(|z| = 1\) as part of the closed contour and indent around the singularity at \(z = 1\), and check L’Hôpital’s rule to see if the singularity is real or “removable” on each side of the branch cut from the log.

11. (Optional, EXTRA CREDIT) The real integral

\[ I = \int_0^{2\pi} \frac{d\theta}{5 + 4 \cos \theta} \]

can be evaluated using

\[ I = \oint_C f(z) \, dz \quad \text{where } C \text{ is the unit circle, } |z| = 1. \]

(a) Use residues to determine the value of the integral.

(b) Determine Taylor or Laurent series for \(f(z)\) in powers of \(z\) for each of the three rings (annular domains) between this function’s two singularities.

Hint: Use partial fractions to write \(f(z)\) as a sum of two first order poles, then use the geometric series appropriately (see Lecture 31 notes).\(^4\)

(c) Show that the series from (b) that is valid on the circle \(|z| = 1\) satisfies \(c_{-n} = c_n\) as expected from the Fourier series \(g(\theta) = \sum_n c_n e^{in\theta}\) since \(g(\theta) = 1/(5 + 4 \cos \theta)\) is a real-valued function.

12. (Optional, EXTRA CREDIT) A complex contour integral formula for the Bessel function of order zero is

\[ J_0(t) = \frac{1}{2\pi i} \oint_C \frac{1}{z} \exp \left( \frac{t}{2} \left[ z - \frac{1}{z} \right] \right) \, dz \]

(a) Show that if the contour \(C\) is taken as the unit circle, \(z = e^{i\theta}\), then this can be reduced to

\[ J_0(t) = \frac{1}{\pi} \int_0^\pi \cos(t \sin \theta) \, d\theta \]

(b) Show that the integral formula from (a) solves Bessel’s equation for \(y(t) = J_0(t)\):

\[ t \frac{d^2y}{dt^2} + \frac{dy}{dt} + ty = 0 \]

Hint: As usual, interchange order of integration and differentiation, and use integration by parts...

Do not spend too much time on the extra credit problems if you have other work to do, or they seem more trouble than they might be worth...

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\(^4\)Products of \(z\) and \(1/z\) series can be difficult to make sure that the result has been completely summed.