Separation of Variables and Eigenfunction expansions for PDE BVP (Part 2)

0. Readings

- Haberman, Section 8.6 and Section 2.5.1.
- Haberman, Sections 7.3, 7.2, 7.4, 7.5
- Skim the Wikipedia article on “Hearing the shape of a drum” – This discusses how the eigenvalues (which give the natural frequencies of vibration, squared) of a Helmholtz equation derived from separation of variables for a wave equation, $u_{tt} = c^2 \nabla^2 u$, with homogeneous Dirichlet boundary conditions are related to the shape of the domain.


   Follow Haberman’s instructions: “Do not reduce to homogeneous BCs.” Namely, use the steps:
   
   i. Use the spatial direction ($x$ or $y$) with homogeneous BC’s to pick the variable for the eigenfunctions, then write the solution as an eigen-expansion in terms of these eigenfunctions times coefficients depending on the other variable.
   
   ii. Then you’ll get ODE BVP’s for each of those coefficient functions. Solve each of those BVP’s in terms of eigen-expansions in that variable.
   
   iii. Your final answer will be a double-sum with constants times $f_n(x)g_m(y)$.
   
   iv. Your final answer should have an explicit formula for the constants with everything worked-out except for a double integral of $Q(x, y)$.


   This time ignore Haberman’s instructions: Do not “reduce to homogeneous BCs” and instead write the solution as a “super-superposition” of two solutions:

   (a) The solution of a Laplace problem with an inhomogeneous Dirichlet BC.
   
   (b) The solution of a Poisson problem with homogeneous Dirichlet BCs.


   Do NOT use “physical reasoning”, instead use the Fredholm alternative theorem for (a) and use the “2-D eigenfunction” solutions of the Helmholtz problem for (b).

   Explain why the solution of part (c) for problem 8.6.10 would have the same result.
   
   (Hint: what is the eigenfunction for $\lambda_0 = 0$?)