Eigenvalue and Boundary Value Problems for Linear ODEs

0. Reading: Haberman, Sections 5.3, 5.5, 5.10.

1. Sturm-Liouville form for all second-order equations – Consider the basic eigenvalue problem
\[ \mathcal{L}\phi = -\lambda \phi \] for general second-order linear operators,
\[ A(x)\frac{d^2 \phi}{dx^2} + B(x)\frac{d\phi}{dx} + C(x)\phi = -\lambda \phi \] (1)
where \( A(x), B(x), C(x) \) are given functions.\(^1\) Equation (1) can always be put into Sturm-Liouville form, \( \mathcal{L}\phi = \phi \),
\[ \frac{d}{dx} \left( p(x) \frac{d\phi}{dx} \right) + q(x)\phi = -\lambda \sigma(x)\phi, \] (2)
even if \( \mathcal{L} \) is not formally self-adjoint in the standard inner product.

(a) Show this by determining \( p(x), q(x), \sigma(x) \) in terms of \( A(x), B(x), C(x) \).

Hint: Expand the product rule term in (2) and multiply the equation across by an unknown integrating factor function, \( m(x) \), and match that equation term by term with (1) to obtain four equations for \( p(x), q(x), \sigma(x), m(x) \).

(b) If you are given the eigenfunctions \( \phi_k(x) \) for the Sturm-Liouville operator \( \mathcal{L} \), how can you write the \( k \) adjoint eigenfunctions for \( \mathcal{L} \)? Hint: Consider the orthogonality relations.

(c) For Sturm-Liouville problems, \( \mathcal{L}u = f(x) \) on \( a \leq x \leq b \) with inhomogeneous boundary conditions, derive Green’s formula (see Haberman 5.5.8), namely evaluate \( \langle \phi_k, \mathcal{L}u \rangle_2 = B_k + \langle \hat{\mathcal{L}}\phi_k, u \rangle_2 \).

2. Reducing Gibbs’ phenomenon – It can be useful to obtain the solution of a BVP in a form that minimizes Gibbs’ phenomenon issues at the boundaries. This question illustrates how to do that by breaking up the solution into two pieces: an eigenfunction expansion for the inhomogeneous forcing and a separate particular solution (not an eigen-expansion) to satisfy the inhomogeneous boundary conditions. The problem
\[ \mathcal{L}u(x) = f(x) \quad BC_1u(a) = c \quad BC_2u(b) = d \] (3)
can be solved in two steps by writing the solution as \( u(x) = u_B(x) + u_F(x) \) where

- The “boundary solution”: \( u_B(x) \) satisfies the unforced equation with the original inhomogeneous boundary conditions
\[ \mathcal{L}u_B(x) = 0 \quad BC_1u_B(a) = c \quad BC_2u_B(b) = d \] (4)

- The “forced solution”: \( u_F(x) \) satisfies the original forced equation with homogeneous boundary conditions
\[ \mathcal{L}u_F(x) = f(x) \quad BC_1u_F(a) = 0 \quad BC_2u_F(b) = 0 \] (5)

Apply this to the example BVP from Lecture 5: \( \frac{d^2u}{dx^2} = 9e^{4x}, \quad u(0) = -5, \quad u(1) = -7 \):

(a) Find \( u_B(x) \) as a polynomial function that solves \( u_B'' = 0, \quad u_B(0) = -5, \quad u_B(1) = -7 \).

(b) Find \( u_F(x) \) as an eigenfunction expansion, \( u_F'' = 9e^{4x}, \quad u_F(0) = 0, \quad u_F(1) = 0 \).

\(^1\)Haberman’s question 5.3.3 is similar. This is also the ODE analogue of the weighted inner product for the non-symmetric matrix, question 5.5A.3, Homework #1 question 3.
(c) Show that your overall solution \( u = u_B + u_F \) matches the Fourier coefficients \( c_k \), equation (22), found for \( u(x) \) in Lecture 5.

(d) The series for \( u_F(x) \) still has some Gibbs’ phenomena in it, but at a “weaker level”. Based on its series coefficients for \( k \to \infty \), how smooth is \( u_F(x) \)? (What is the lowest order derivative of \( u_F(x) \) that is discontinuous?)

3. Consider the eigenvalue problem on \( 0 \leq x \leq \pi \),

\[
\frac{d^2 \phi}{dx^2} + 8 \frac{d\phi}{dx} + (16 + \lambda)\phi = 0, \\
\phi'(0) + 4\phi(0) = 0, \quad \phi'(\pi) + 4\phi(\pi) = 0.
\]

(a) Assuming \( \lambda \) to be a non-negative constant (consider the cases \( \lambda = 0 \) and \( \lambda > 0 \)), what is the general solution of the homogeneous ODE.

(b) Apply the boundary conditions to determine the eigenvalues and eigenfunctions.

(c) What is the adjoint problem? Obtain the adjoint eigenfunctions.

(d) What are the functions \( p, q, \sigma \) that put this problem in standard Sturm-Liouville form? (see Problem 1)

(e) Using (a) and (b) write the simplest forms for the integrals for the standard inner product \( \langle \phi_k, \psi_j \rangle_2 \) and the weighted inner product \( \langle \phi_k, \phi_j \rangle_\sigma \) orthogonality conditions and show that they are equivalent to each other.

(f) Obtain the coefficients \( c_k \) in the expansion of the function \( u(x) = \sum_{k=0}^{\infty} c_k \phi_k(x) \) that solves the boundary value problem,

\[
\frac{d^2 u}{dx^2} + 8 \frac{du}{dx} + 12u = 9, \\
u'(0) + 4u(0) = 5 \quad u'(\pi) + 4u(\pi) = 3.
\]

Hint: If \( \phi(x) \) is a solution of \( L\phi = -\lambda \phi \) then it is also a solution of \( L\phi - c\phi = -\tilde{\lambda} \phi \) where \( \tilde{\lambda} = \lambda + c \).

4. Another inhomogeneous boundary value problem:

(a) Find the general homogeneous solution of the Cauchy-Euler equation

\[
x^2 \frac{d^2 \phi}{dx^2} + 5x \frac{d\phi}{dx} + (4 + \alpha)\phi = 0,
\]

where \( \alpha \) is assumed to be a positive constant.

(b) Use (a) to determine the eigenvalues and eigenfunctions of the Sturm-Liouville problem on \( 1 \leq x \leq e \)

\[
\frac{d}{dx} \left( x^5 \frac{d\phi}{dx} \right) + \lambda x^3 \phi = 0 \quad \phi(1) = 0 \quad \phi(e) = 0.
\]

(c) Use (b) to obtain the eigenfunction expansion for the solution of the inhomogeneous problem

\[
\frac{d}{dx} \left( x^5 \frac{du}{dx} \right) = -2x^2 \quad u(1) = 3 \quad u(e) = 7.
\]