

Math 224: Scientific Computing I

Homework 8

Assigned: Monday, Nov 8, 2004

Due: Friday, Nov 19, 2004

QR Factorization and Least Squares Problems

0. Reading: Trangenstein Chapter 6, Numerical Recipes 2.9, 2.10, Atkinson 8.3, 9.3.

1. [New Routines] Write routines for:

(a) The classic Gram-Schmidt (CGS) construction of the QR factorization of a matrix $A_{m \times n} = Q_{m \times n} R_{n \times n}$

```
int QRfactorCGS(double **A,int m,int n,double **Q,double **R);
```

Return 1 if successful, return 0 if the factorization fails because the columns are not linearly independent.

(b) The modified Gram-Schmidt (MGS) construction of the QR factorization of a matrix $A_{m \times n} = Q_{m \times n} R_{n \times n}$

```
int QRfactorMGS(double **A,int m,int n,double **Q,double **R);
```

(c) The solution of $A\vec{x} = \vec{b}$ given by the QR factorization, where $\vec{x} \in \mathbb{R}^n$ and $\vec{b} \in \mathbb{R}^m$

```
void QRsolve(double **Q,double **R,int m,int n,double *b,double *x);
```

Put the solution in \vec{x} .

(d) Solution of the Normal equations $A^T A \vec{x} = A^T \vec{b}$ using LU factorization

```
void Normalsolve(double **A,int m,int n,double *b,double *x);
```

You may either write a Cholesky factorization routine or use your earlier LU routines.

(e) Solution of $A\vec{x} = \vec{b}$ using Householder reflectors

```
void Householder(double **A,int m,int n,double *b,double *x);
```

2. [Testing the routines: X-matrix³, the finale] Let A be a $n \times n$ matrix (n must be an **EVEN** integer), whose entries are all zeroes, except for:

$$a_{i,i} = i \quad a_{i,n-i+1} = i/2, \quad i = 1, 2, \dots, n$$

and $b_i = 1$ for $i = 1, 2, \dots, n$. For $n = 6, 8, 10, \dots, 500$:

(a) Calculate the solution of $A\vec{x} = \vec{b}$ using your earlier LU routines. Consider this solution to be the “exact” answer, \vec{x}_* .

(b) Find the max-norm error, $e_n = \|\vec{x} - \vec{x}_*\|_\infty$, between the exact solution and the solution calculated using each of the following methods:

- QRfactorCGS()/QRsolve()
- QRfactorMGS()/QRsolve()
- Normalsolve()
- Householder()

Plot the error curves for each vs n together on one graph and conclude which is the best/worst of the methods for square systems.

3. [Solving Least Squares Problems: The \triangleright -matrix: revenge of X] Let A be a $m \times (m/2)$ matrix (m must be an **EVEN** integer), whose entries are all zeroes, except for:

$$a_{i,i} = i \quad a_{m-i+1,i} = (m-i+1)/2 \quad i = 1, 2, \dots, m/2, \quad a_{i,1} = \frac{m+i(m-2)}{2m-2}, \quad i = 1, 2, \dots, m$$

and $b_i = 1$ for $i = 1, 2, \dots, m$. This matrix is similar to “the left half” of the X matrix. Find the least squares solution of $A\vec{x} = \vec{b}$, where $\vec{x} \in \mathbb{R}^{m/2}$. Use the method judged “best” from the previous problem as giving the “exact” answer here and compare the other methods for LS problems as was done in the previous problem (i.e. plot of error vs n).

What is \vec{x}_{LS} for $m = 10$? Give to as many digits as possible.