

## Math 211, Fall 2009, Test 2

1. (20 pts) Consider the inhomogeneous boundary value problem for  $u(x)$  on  $0 \leq x \leq 1$ :

$$\frac{d^2 u}{dx^2} - 2 \frac{du}{dx} + u = f(x), \quad u(0) = 4 \quad u(1) = -7$$

State the jump conditions and determine the piecewise-defined Green's function for this problem. (DO NOT solve for  $u(x)$ .)

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2. (25 pts) Consider the problem for  $u(r, t)$  on  $0 \leq r \leq 1$  and  $t \geq 0$ :

$$\frac{\partial u}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + Q(r) \cos t$$

$$u(r, 0) = e^r \quad \text{for } 0 \leq r \leq 1 \quad \text{and} \quad \left. \frac{\partial u}{\partial r} \right|_{r=1} = \cos t \quad \text{for } t \geq 0$$

The solution can be written in the form

$$u(x, t) = \sum_{n=0}^{\infty} b_n(t) J_0(\sqrt{\lambda_n} r)$$

- (a) (4 pts) Explain what determines the  $\lambda_n$  values.
- (b) (16 pts) Determine the ODEs and initial conditions for the  $b_n(t)$ 's for  $n = 0, 1, 2, \dots$ . Simplify as much as possible. (DO NOT solve these equations.)
- (c) (5 pts) What condition on the problem must hold in order for the solution to have the integral  $\int_0^1 u(r, t) r dr$  be constant for all times? Simplify your answer as much as possible.
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3. (55 pts) Consider the PDE for  $u(x, y)$  on the rectangle  $\{ 1 \leq x \leq e \text{ and } 0 \leq y \leq \pi \}$ :

$$x^2 \frac{\partial^2 u}{\partial x^2} + x \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial y^2} = Q(x, y)$$

$$u(x, 0) = 0 \quad u(x, \pi) = 0 \quad \text{for } 1 \leq x \leq e$$

$$u(1, y) = 0 \quad u(e, y) = 0 \quad \text{for } 0 \leq y \leq \pi$$

The solution to this problem can be written in the form

$$u(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} c_{nm} f_n(x) g_m(y)$$

If  $\phi_{nm}(x, y) = f_n(x) g_m(y)$  is an eigenfunction of the boundary value problem for the linear operator  $L\phi \equiv x^2 \phi_{xx} + x \phi_x + \phi_{yy}$  (i.e.  $L\phi_{nm} = -\lambda_{nm} \phi_{nm}$ ) then

- (a) (20 pts) Use separation of variables to determine  $f_n(x)$  and  $g_m(y)$  and the eigenvalues  $\lambda_{nm}$ .
- (b) (7 pts) Write the ODE for  $f_n(x)$  in self-adjoint form by dividing across by an appropriate power of  $x$ . Use this to determine the appropriate inner product for this problem.
- (c) (12 pts) Determine the  $c_{nm}$  coefficients in terms of integrals with  $f, g, Q$ . Simplify your answer as much as possible.
- (d) (16 pts) If there were no source ( $Q \equiv 0$ ) but there was an inhomogeneous boundary condition,  $u(x, \pi) = A(x)$ , determine the  $c_{nm}$  coefficients in terms of integrals with  $f, g, A$ . Simplify your answer as much as possible.
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