

# Math 211, Fall 2009, Test 1

1. (50 pts) You are given that for any value of the constant  $\alpha$ , the differential equation

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + \left(\alpha x^2 - \frac{1}{4}\right) y = 0 \quad \text{has general solution} \quad y(x) = c_1 \frac{\sin(\sqrt{\alpha} x)}{\sqrt{x}} + c_2 \frac{\cos(\sqrt{\alpha} x)}{\sqrt{x}}.$$

Consider bounded solutions of the eigenvalue problem for  $\phi(x)$  on  $0 \leq x \leq 1$ :

$$x^2 \frac{d^2 \phi}{dx^2} + x \frac{d\phi}{dx} - \frac{1}{4} \phi = -\lambda x^2 \phi, \quad \phi(1) = 0. \quad (1)$$

- (a) (7 pts) Determine the eigenvalues and eigenfunctions.
- (b) (8 pts) Determine the adjoint operator and adjoint boundary conditions.
- (c) (3 pts) Determine  $p(x), q(x), \sigma(x)$  that put the problem into the form of a singular Sturm-Liouville equation,  $(p(x)\phi')' + q(x)\phi = -\lambda\sigma(x)\phi$ .  
Hint: Divide the equation across by  $x$ .
- (d) (3 pts) Use (a,c) to write the integral in simplest form for the orthogonality relation between  $\phi_j$  and  $\phi_k$  with respect to  $\sigma(x)$  for  $j \neq k$ . (DO NOT evaluate the integral.)
- (e) (5 pts) For a function  $f(x)$  given on  $0 \leq x \leq 1$ , write the formula for the coefficients in its expansion,  $f(x) = \sum_k f_k \phi_k(x)$ , in terms of integrals in simplest form with the eigenfunctions from (a).
- (f) (24 pts) Find the coefficients in the eigenfunction expansion of the solution,  $u(x) = \sum_k c_k \phi_k(x)$ , of the boundary value problem

$$x^2 \frac{d^2 u}{dx^2} + x \frac{du}{dx} - \frac{1}{4} u = x^{5/2}, \quad u(1) = 1.$$

Simplify your answer as much as possible.

Hint: First, divide the problem across by  $x$  to make the LHS match Sturm-Liouville form, and then solve that equation.

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2. (50 pts) Consider the linear operator

$$Lu \equiv \int_0^1 [16x - 30x^2 t^2] u(t) dt. \quad (2)$$

- (a) (17 pts) Determine all of the eigenvalues and determine the eigenfunctions for the eigenvalues of finite multiplicity in  $L\phi = \lambda\phi$ .
- (b) (12 pts) Write the adjoint operator. Determine the adjoint eigenfunctions for the eigenvalues of finite multiplicity.
- (c) (21 pts) Let  $\beta$  be a real constant. Consider the second kind Fredholm integral equation for  $u(x)$ ,

$$Lu + \beta u(x) = 1 - 8x^3.$$

- i. (3 pts) For what value(s) of  $\beta$  does this equation have no solution?
- ii. (3 pts) For what value(s) of  $\beta$  does this equation have only one solution?
- iii. (3 pts) For what value(s) of  $\beta$  does this equation have infinity many solutions?
- iv. (12 pts) Find a solution for  $\beta = 1$ .

Note: Part (iv) can be done independently of parts (i–iii).

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