Math 211, Fall 2008, Test 2

1. (17 pts) Consider the inhomogeneous boundary value problem for $u(x)$ on $0 \le x \le 1$:

$$\frac{d^2u}{dx^2} - 2\frac{du}{dx} = f(x), \qquad u(0) = 2 \qquad u'(1) = -1$$

Give the piecewise-defined form for the Green's function for this problem.
(DO NOT solve the full problem.)

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2. (25 pts) Consider the problem for $u(x,t)$ on $0 \le x \le \pi$ and $t \ge 0$:

$$u_t = u_{xx} + 4u - e^t \sin x$$

$$u(x, t=0) = \sin(2x) \qquad \text{for } 0 \le x \le \pi$$

$$u(x=0,t) = \cos t \qquad u(x=\pi,t) = t \qquad \text{for } t \ge 0$$

The solution can be written in the form

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t) \sin(nx)$$

Determine the ODEs and initial conditions for the $b_n(t)$'s in simplest form.
(DO NOT solve these equations.)

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3. (23 pts) Consider the Helmholtz eigenvalue equation $\nabla^2\phi = -\lambda\phi$ on the rectangular domain $0 \le x \le L$ and $0 \le y \le H$ subject to the boundary conditions

$$\left.\frac{\partial\phi}{\partial y}\right|_{y=0} = 0 \qquad \left.\frac{\partial\phi}{\partial y}\right|_{y=H} = 0 \qquad \text{for } 0 \le x \le L$$

$$\phi(x=0,y) = 0 \qquad \left.\frac{\partial\phi}{\partial x}\right|_{x=L} = 0 \qquad \text{for } 0 \le y \le H$$

(a) (10 pts) Determine the eigenvalues and eigenfunctions.

(b) (13 pts) Use (a) to write the solution $u(x,y)$ to the Poisson problem $\nabla^2 u = f(x,y)$ with the same domain and the same homogeneous boundary conditions. Simplify as much as possible.

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4. (35 pts) Consider Laplace's equation for u on the domain $1 \le r \le 2$ and $0 \le \theta \le \pi$ subject to the boundary conditions

$$u(r,\theta=0) = 0 \qquad \left.\frac{\partial u}{\partial\theta}\right|_{\theta=\pi} = 0 \qquad \text{for } 1 \le r \le 2$$

$$u(r=1,\theta) = f(\theta) \qquad \left.\frac{\partial u}{\partial r}\right|_{r=2} = g(\theta) \qquad \text{for } 0 \le \theta \le \pi$$

The solution can be written in the form

$$u(r,\theta) = \sum_{n=1}^{\infty} a_n(r) \sin(\sqrt{\lambda_n}\,\theta)$$

(a) (5 pts) Determine the λ_n's.

(b) (12 pts) Write the ODEs and boundary conditions for the $a_n(r)$ functions.

(c) (18 pts) Solve for the $a_n(r)$'s for

$$f(\theta) = \cos\theta, \qquad g(\theta) = 1.$$
