

# 1 Math 211, Fall 2008, Test 1

1. (40 pts) Consider the linear operator

$$L\phi \equiv \phi(x) + \int_0^\infty [12e^{-2x-t} - 30e^{-x-2t}] \phi(t) dt. \quad (1)$$

- (a) (3) Determine the adjoint operator.  
(b) (17) Determine all of the eigenvalues for  $L\phi = \lambda\phi$ . Determine the eigenfunctions for the eigenvalues of finite multiplicity.  
(c) (10) Determine the adjoint eigenfunctions for the eigenvalues of finite multiplicity.  
(d) (10) Consider the inhomogeneous problem for  $u(x)$  on  $0 \leq x < \infty$

$$Lu = ce^{-2x} + 6e^{-x}$$

for all values of the parameter  $c$ . For which values of  $c$  is the solution unique? For which values of  $c$  is there no solution? For which values of  $c$  do multiple solutions exist?

2. (60 pts) Consider the following eigenvalue problem for  $\phi(x)$  on  $0 \leq x \leq 1$ :

$$\frac{d^2\phi}{dx^2} + \phi = -\lambda\phi, \quad \phi(0) = 0, \quad \left. \frac{d\phi}{dx} \right|_{x=1} = 0. \quad (2)$$

- (a) (8) Calculate the adjoint operator and adjoint boundary conditions. Show that the problem is self-adjoint. What can you conclude about: (i) the eigenvalues? (ii) the inner products of different eigenfunctions?  
(b) (10) Find the eigenvalues  $\lambda_k$  and eigenfunctions  $\phi_k(x)$ .  
(c) (14) For a typical (nice) function  $f(x)$  given on  $0 \leq x \leq 1$ ,  
i. Write the formulas for the coefficients in its expansion in terms of the eigenfunctions from (b),  $f(x) = \sum_k f_k \phi_k(x)$ .  
ii. Suppose that  $f(x) = x^6$  on  $0 \leq x \leq 1$ , then  
(I) Sketch the graph that would be generated by your eigenfunction expansion on  $-2 \leq x \leq 2$  and  
(II) State the limiting power  $\alpha$  that the coefficients should have for  $k \rightarrow \infty$  (i.e.  $f_k \rightarrow \beta/k^\alpha$ ) (DO NOT calculate the coefficients).  
(d) (6) Consider the inhomogeneous problem

$$\frac{d^2u}{dx^2} + u = g(x), \quad u(0) = a_0, \quad \left. \frac{du}{dx} \right|_{x=1} = a_1,$$

where  $g(x), a_0, a_1$  are given. Explain and justify how you can deduce from part (b) that the solution to this problem must be unique.

- (e) (22) Find the coefficients in the eigenfunction expansion

$$u(x) = \sum_k c_k \phi_k(x)$$

that solves

$$\frac{d^2u}{dx^2} + u = \pi^3 x, \quad u(0) = \frac{1}{2}, \quad \left. \frac{du}{dx} \right|_{x=1} = \pi.$$

Simplify your answer as much as possible.

---