

This is a method for solving Linear Constant Coefficient (LCC) DE's.¹

1) Put the problem in standard form:²

$$y'' + py' + qy = f_0(x) + f_1(x) + \dots$$

For a 2nd order LCCDE, the **general** solution has **two** constants of integration.

For a solution of a specific **initial (or boundary) value problem** for a 2nd order LCCDE, you will be given **two** conditions that you can use to solve for the **two** unknown constants.

If RHS = 0 then the DE is homogeneous (unforced) and you only need the homogeneous solution (step 2),

$$y(x) = y_h(x)$$

If RHS $\neq 0$ then the DE is called inhomogeneous and the general solution is the sum of solutions from step 2 and step 3,

$$y(x) = y_h(x) + y_p(x)$$

where $y_p(x)$ is the “particular” (forced) solution, and it includes contributions from each of the RHS $f(x)$ terms. Each of the $f(x)$ terms on the RHS must look like

$$f(x) = P(x)e^{Rx} = (\text{polynomial}) \cdot (\text{exponential})$$

Since R can be complex, the exponential can be a complex exponential with sine and/or cosine terms.

2) **FIRST, ignore the RHS**, solve the homogeneous DE (RHS=0):

$$y'' + py' + qy = 0$$

Guess the trial solution

$$y(x) = e^{rx}$$

You can use the quick substitution rule:³

$$y \rightarrow r^0 = 1 \quad y' \rightarrow r^1 = r \quad y'' \rightarrow r^2 \quad \dots$$

to get the characteristic polynomial equation

$$r^2 + pr + q = 0$$

Get the roots r_1 and r_2 .⁴

The DE will have one solution⁵ for each root. The first solution is always

$$y_1(x) = e^{r_1x}$$

If $r_2 \neq r_1$, then the other solution of the DE is

$$y_2(x) = e^{r_2x}$$

If $r_2 = r_1$ (a double root), then the second solution for r_1 is $y_1(x)$ multiplied by x ,

$$y_2(x) = xy_1(x) = xe^{r_1x}$$

If r_1 is complex, $r_1 = a + ib$, then

$$y_1(x) = e^{ax}e^{ibx}$$

The general solution of the homogeneous DE is the sum of these solutions, each multiplied by a different constant

$$y_h(x) = c_1y_1(x) + c_2y_2(x)$$

For the complex case, by playing with the constants, all solutions can be written as being real

$$y_h(x) = c_3e^{ax} \cos bx + c_4e^{ax} \sin bx$$

or

$$y_h(x) = Ae^{ax} \cos (bx - \phi)$$

3) **NOW, mostly ignore the LHS**, guess the “particular forcing” solutions. There will be one $y_p(x)$ solution from each $f(x)$ term on the RHS and the final particular solution will be the sum

$$y_p(x) = y_{p0}(x) + y_{p1}(x) + \dots$$

The final $y_p(x)$ solution will not have any unknown constants. To figure out the proper form of $y_p(x)$, look at $f(x)$ and let

$$y_p(x) = Q(x)x^m e^{Rx}$$

R in $y_p(x)$ is the same as R in $f(x)$

$$Q(x) = Ax^n + Bx^{n-1} + Cx^{n-2} + \dots + Z$$

is a polynomial of the same size as $P(x)$ in $f(x)$, but you will have to figure out all of the unknown coefficients

What's x^m ? If R from $f(x)$ is the same number as one of the roots of the characteristic polynomial from the LHS, then you need to multiply the “usual” $y_p(x)$ up by x (just like with a double root). If R matches m roots then multiply up by x^m . Example $R = r_1 = r_2$, then $m = 2$. If R is not r_1 or r_2 then $m = 0$. If R is complex, then guess

$$y_p(x) = Q_1(x)x^m e^{ax} \cos (bx) + Q_2(x)x^m e^{ax} \sin (bx)$$

After that, plug $y_p(x)$ into the original LCCDE and grunge through the algebra until you determine all of the coefficients in $Q(x)$.

Notes

¹This trick works for 1st, 2nd, ... any-th order LCCDE's, but 2nd order is most common.

²Standard form is (a sum of constants times derivatives of $y(x)$ on the LHS) equals (a sum of functions $f(x)$ on the RHS).

³, (to the order of the derivative of $y(x)$)

⁴Factor the polynomial or use the quadratic formula.

⁵one distinct “linearly independent” solution for each root