

1 Integrating First Order Ordinary Differential Equations

1. Integration: If you are solving a specific initial value problem, use definite integrals. If you are finding a general solution of an ODE, use indefinite integrals. When calculating an integrating factor, use an indefinite integral and drop the “+C.” If you get confused by using too many different letters for dummy variables of integration, use CAPITALS:

$$\frac{dy}{dx} = g(x) \quad y(x_0) = y_0 \quad \rightarrow \quad y(x) = \int_{x_0}^x g(X) dX + y_0$$

Review the various methods of integration: **partial fractions, trigonometric substitutions, integration by parts, etc.**

2. Deriving ODE's: Use the given assumptions appropriate for the science in the problem – chemistry, physics, etc. Remember that “rate of change of $y(t)$ ” means dy/dt . For geometry, remember that the derivative is the slope, $dy/dx = \tan \theta$.

3. Determine the appropriate way to solve the problem by identifying the type of the ODE:

First, put the ODE in **standard form**:¹ ²

$$\frac{dy}{dx} = H(x, y)$$

then check the type of ODE:

a) Is it **simple**?

$$\frac{dy}{dx} = f(x)$$

Just integrate.

b) Is it **separable**?

$$\frac{dy}{dx} = \frac{f(x)}{g(y)}$$

Separate x 's and y 's and integrate.³

c) Is it **linear**?⁴

$$\frac{dy}{dx} + P(x)y = Q(x)$$

Calculate integrating factor $\rho(x) = e^{\int P(x) dx}$ then multiply the ODE across by $\rho(x)$, let $w = \rho(x)y$ and integrate the resulting **simple** ODE for $w(x)$:

$$\frac{dw}{dx} = \rho(x)Q(x)$$

If these don't work, try:

d) Is it **Bernoulli**?

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

Let $w = y^{1-n}$ and multiply the ODE to get a **linear** ODE for $w(x)$.⁵

e) Is it **homogeneous**?⁶

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right)$$

Let $y = vx$ to get a **separable** ODE for $v(x)$.

If these don't work, try:

f) Is it **upside-down**?

$$\frac{dy}{dx} = \text{MESS} \quad \text{but} \quad \frac{dx}{dy} = \text{NICE} = \frac{1}{\text{MESS}}$$

Integrate the easier ODE for $x(y)$ to get an implicit solution. For **homogeneous** ODE, let $x = uy$.

g) Does the ODE look like it came from a **product rule** or **chain rule** for a derivative? – Try to undo it!

h) **Fancy substitutions**: If you see some “interesting” combination of y 's and x 's inside a \exp , \sin , \cos , or etc, try to let w = “the combination of variables.”

i) **Existence and Uniqueness**: Does a theorem say that no solution exists? or more than one solution exists? Does $\frac{\partial H}{\partial y}$ blow up?

Some ODE's fit into several types, they can be solved in several different ways. Try these:

1) Use (b), (e) for

$$\frac{dy}{dx} = \frac{y}{x}$$

2) Use (d), (e) for

$$\frac{dy}{dx} + \frac{y}{x} = \frac{y^3}{x^3}$$

3) Use (e), (f) for

$$\frac{dy}{dx} = -\frac{2x^3y}{x^4 + y^4}$$

4) Use (h) for

$$\frac{dy}{dx} - \frac{4}{x}y = 2x^5e^{y/x^4}$$

Notes

¹ Standard form means: no factors multiplying the derivative, all else on RHS.

² These techniques are arranged in order of (usually) increasing difficulty to integrate, so try to use the first method that applies to your ODE.

³ Formally true, but recall full details: reduce (b) to a **simple** ODE (a) by multiplying across by $g(y)$ and using the chain rule to get the ODE for $w(x)$:

$$w = \int g(y) dy = G(y) \quad \rightarrow \quad \frac{dw}{dx} = f(x)$$

⁴ linear in $y(x)$.

⁵ The **Bernoulli** approach doesn't work for $n = 1$ (**separable** (b)) or $n = 0$ (**linear** (c)), but you already know how to solve these cases.

⁶ A good way to test if the ODE is **homogeneous** is to plug in $y = vx$ and see if the right-side is $F(v)$ (no y 's or x 's alone).

2 Basic algebra of Complex Numbers

Standard form: a and b are real numbers⁷

$$z = a + ib = \operatorname{Re}(z) + i \operatorname{Im}(z)$$

Complex conjugate

$$\bar{z} = a - ib$$

Absolute value⁸

$$|z| = \sqrt{a^2 + b^2}$$

$$|z|^2 = a^2 + b^2 = z\bar{z}$$

All about i :⁹ $i = \sqrt{-1}$

$$\bar{i} = -i \quad i^2 = -1 \quad i^3 = -i \quad i^4 = 1$$

$$\operatorname{Re}(z) = \frac{z + \bar{z}}{2} \quad \operatorname{Im}(z) = \frac{z - \bar{z}}{2i}$$

Euler's equation¹⁰

$$e^{i\theta} = \cos \theta + i \sin \theta$$

$$e^{2\pi i} = 1 \quad 1^k = 1 \quad 1 = e^{i2\pi k}$$

Deriving trig identities:

$$\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$e^{in\theta} = (e^{i\theta})^n$$

$$\cos(n\theta) + i \sin(n\theta) = (\cos \theta + i \sin \theta)^n$$

Expand¹¹ the right-side and separate Re and Im parts. Ex: $\cos(2\theta) = \operatorname{Re}((\cos \theta + i \sin \theta)^2)$.

Rectangular and Polar forms

$$z = a + ib = re^{i\theta} \quad \bar{z} = a - ib = re^{-i\theta}$$

Converting rectangular to polar

$$r = |z| = \sqrt{a^2 + b^2} \geq 0 \quad \theta = \tan^{-1}(b/a)$$

Converting polar to rectangular

$$a = r \cos \theta \quad b = r \sin \theta$$

Division

$$\frac{1}{z} = \frac{\bar{z}}{|z|^2} = \frac{1}{r} e^{-i\theta} = \frac{a - ib}{a^2 + b^2}$$

What are they good for?

Polar: multiply, divide, powers, "easy angles"¹²."

Rectangular: add, subtract, separate Re and Im parts, "hard angles"¹³."

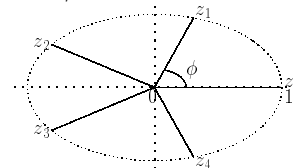
Solve $z^n = 1$ really means:

$$\text{solve } z^n = e^{i2\pi k}$$

Let $\phi = 2\pi/n$, then take **(both sides)**^{1/n} and the solutions are

$$z_k = e^{ik\phi} \quad \text{with } k = 0, 1, 2, \dots, (n-1)$$

Ex: $n = 5$, so $\phi = 2\pi/5 = 72^\circ$.

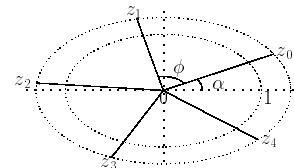


Solve $z^n = a + ib$ really means convert $w = a + ib$ to polar $w = re^{i\theta}$ and solve

$$z^n = re^{i\theta} e^{2\pi ki} = re^{i(\theta + 2\pi k)}$$

taking **(both sides)**^{1/n}, let $\alpha = \theta/n$, then

$$z_k = r^{1/n} e^{i(\alpha + k\phi)}$$



so the roots get *stretched* by $r^{1/n}$ and *rotated* by angle α . Finally, solving $(z - c)^n = a + ib$ gives solutions that are *translated* by c : $z_k = c + r^{1/n} e^{i(\alpha + k\phi)}$.

Notes

⁷ a is called the REAL part of z , b is called the IMAGINARY part of z .

⁸Also called the length or modulus.

⁹Odd powers are imaginary, Even powers are real.

¹⁰ $k = 0, \pm 1, \pm 2, \dots$ (k can be any integer).

¹¹Remember the binomial expansion?

¹²Angles like $30^\circ, 45^\circ, 60^\circ, 90^\circ, \dots$

¹³Angles where you'd have to look up sin and cos.