

Math 211, Fall 2009, Duke University

Strategy

- Identify the type of integral you have.
- Try using the simplest technique for your integral.
- If that doesn't work, try the next-level method (might involve more work, but is more likely to work).
- You may have to break-up the integral into several parts and use a different method for each part.

u-Substitution

Goal: Reduce the integral to one of the 10 basic integrals in the table.

Works: If the integrand is nice enough to have exactly the chain rule form: $\int f'(g(x))g'(x) dx = \int f'(u) du = f(g(x)) + C$

Examples: $\int 2x \cos(x^2) dx = \int \cos(u) du$

Integration by parts

Goal: Figure out the integral of a product of two functions (polynomials, exponentials, logs, trig, inverse trig, etc).

Works: For all the times when u -substitution fails

Notes: Needed for secant^{oddpower} (Use Trig-Tricks for most other trig integrals)

Examples: $\int x(3x+4)^{777} dx, \int e^x \sin x dx, \int \ln x \tan^{-1} x dx$

Trig Tricks

Goal: Integrals of combinations of trig functions

Examples: $\int \cos^2(3x) \sin(4x) dx$

The 4 steps

Check these steps (in order!) to simplify your integral as much as needed:

- 1: $\alpha \neq \beta$ If you have an integral of a product of trig functions with different angles (Ex: $3x \neq 4x$) you **MUST** use trig identities to break it up into a sum of different terms before you can make any progress:
 $\int \cos(3x) \sin(4x) dx = \frac{1}{2} \int \sin(7x) + \sin(x) dx$

- 2: $\boxed{\text{If one is odd, } u = \text{other}}$ This gives the right u -substitution for $\cos^{\text{power}_1} \times \sin^{\text{power}_2}$ integrals (with positive or zero powers). You will also need to use $\cos^2 x + \sin^2 x = 1$. Example:
 $\int \cos^7 x \sin^2 x dx$ then \cos^{odd} , so $u = \sin x$ and use $\cos^2 x = 1 - u^2$ to get $\int (1 - u^2)^3 u^2 du$

Review of techniques of integration

- 3: $\boxed{\text{even} \times \text{even} \rightarrow \frac{1}{2} \triangleleft}$ If both powers are even, then you need to use the half-angle formulas:

$$\cos^2 x = (1 + \cos(2x))/2$$

$$\sin^2 x = (1 - \cos(2x))/2$$

- 4: $\boxed{\text{SET TOS}}$ Use $\sec^2 x = 1 + \tan^2 x$ with

- SET: $\text{Sec}^{\text{Even}}, u = \text{Tan}$

- TOS: $\text{Tan}^{\text{Odd}}, u = \text{Sec}$

Ex: $\int \sec^4 x \tan^{\sqrt{2}} x dx, \int \sec^{\pi}(6x) \tan^3(6x) dx$

Square roots

- If you have a $\sqrt{ax+b}$ then let $u = \sqrt{ax+b}$ so $u^2 = ax+b$ and $2u du = a dx$ and $x = (u^2 - b)/a$
- If you have a $\sqrt{a^2 - x^2}$ then $\boxed{x = a \sin \theta}$
- If you have a $\sqrt{a^2 + x^2}$ then $\boxed{x = a \tan \theta}$
- If you have a $\sqrt{x^2 - a^2}$ then $\boxed{x = a \sec \theta}$
- If you have a $\sqrt{ax^2 + bx + c}$ then **First, complete the square, then let $u = x - d$ and look again**, example:

$$\sqrt{x^2 - 4x + 7} = \sqrt{(x-2)^2 + \sqrt{3}^2} = \sqrt{u^2 + k^2}$$

- These work with positive or negative powers of the roots and (quadratic) ^{$\pm k$} too.

Rational Functions

“Rational fcn” means $\frac{\text{polynomial}}{\text{polynomial}}$

Easy ones:

- $\int \text{poly} dx = \text{new poly} + C$
- $\int \frac{d\text{poly}}{\text{poly}} = \ln |\text{poly}| + C$
- $\int \frac{dx}{\text{quadratic}}$: Complete the square and use square roots guide.
- $\int \frac{d\text{poly}}{\text{poly}^n} = -\frac{1}{n-1} (\text{poly})^{-n+1} + C \quad n \neq 1$

Everything else needs **Partial Fractions**:

1. NO Improper fractions: Convert $\frac{21}{5} \rightarrow 4 + \frac{1}{5}$. You **MUST** do this first! (Use long division or synthetic division)
2. Factor numerator, denominator, cancel stuff
3. Expand out as a sum of partial fractions (each will be one of the “easy ones”)