1 The full Fourier series

For a given L^2 function $f(x)$ on $-L \le x \le L$

$$f(x) \text{ “=” } \sum_{k=0}^{\infty} a_k \cos\left(\tfrac{k\pi}{L}x\right) + \sum_{k=1}^{\infty} b_k \sin\left(\tfrac{k\pi}{L}x\right), \qquad (= \text{a.e.}) \tag{1a}$$

where the Fourier coefficients are given by the usual expansion formula for self-adjoint orthogonal eigenfunctions applied to the cosines,

$$a_k = \frac{\langle f(x), \cos(k\pi x/L)\rangle}{||\cos(k\pi x/L)||^2} = \begin{cases} \dfrac{1}{2L}\displaystyle\int_{-L}^{L} f(x)\,dx & k=0 \\ \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)\cos(k\pi x/L)\,dx & k=1,2,3,\cdots \end{cases} \tag{1b}$$

and applied to the sines,

$$b_k = \frac{\langle f(x), \sin(k\pi x/L)\rangle}{||\sin(k\pi x/L)||^2} = \frac{1}{L}\int_{-L}^{L} f(x)\sin(k\pi x/L)\,dx. \tag{1c}$$

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2 Connections to Even/Odd functions

The relations between the values for functions for negative vs. positive values of x:

- Even function: $f(-x) = f(x)$
- Odd function: $f(-x) = -f(x)$
- Every function can be broken up into a sum of even and odd parts

$$f(x) = f_{\text{even}}(x) + f_{\text{odd}}(x), \qquad f_{\text{even}}(x) = \frac{f(x)+f(-x)}{2} \qquad f_{\text{odd}}(x) = \frac{f(x)-f(-x)}{2} \tag{2}$$

- The above three items apply if you are given a $f(x)$ defined on a symmetric interval, $-L \le x \le L$. If you are just given $f(x)$ on the positive half-interval, $0 \le x \le L$, you can use these to define the even or odd extensions on the negative interval (the values $f(x)$ should take for $-L \le x \le 0$).
- Integration on symmetric domains

$$\int_{-L}^{L} \text{odd}(x)\,dx = 0 \qquad \int_{-L}^{L} \text{even}(x)\,dx = 2\int_{0}^{L} \text{even}(x)\,dx. \tag{3}$$

- $\cos(kx)$ is even, $\sin(kx)$ is odd
- Products of functions

$$\text{odd}(x)\cdot\text{odd}(x) = \text{even}(x) \qquad \text{even}(x)\cdot\text{odd}(x) = \text{odd}(x) \qquad \text{even}(x)\cdot\text{even}(x) = \text{even}(x) \tag{4}$$

- If $f(x)$ is odd, then by $(4)_2$ and $(3)_1$ $a_k = 0$ (and b_k is doubled from the half-interval value by $(3)_2$) and the full Fourier series reduces down to the Fourier sine series,

$$f(x) = \sum_{k=1}^{\infty} b_k \sin\left(\tfrac{k\pi}{L}x\right) \qquad b_k = \frac{2}{L}\int_0^L f(x)\sin(k\pi x/L)\,dx. \tag{5}$$

  If $f(x)$ is even, then $b_k = 0$ (and a_k is doubled from the half-interval value) and the full Fourier series reduces down to the Fourier cosine series,

$$f(x) = \sum_{k=0}^{\infty} a_k \cos\left(\tfrac{k\pi}{L}x\right) \qquad a_k = \frac{2}{L}\int_0^L f(x)\cos(k\pi x/L)\,dx \qquad a_0 = \frac{1}{L}\int_0^L f(x)\,dx. \tag{6}$$

3 Useful integration facts for calculating Fourier coefficients

- Commonly occurring special values, $\cos(0) = 1$, $\sin(0) = 0$,

$$\cos(k\pi) = (-1)^k \qquad \sin(k\pi) = 0 \qquad \cos\left(\frac{(2k+1)\pi}{2}\right) = 0 \qquad \sin\left(\frac{(2k+1)\pi}{2}\right) = (-1)^k$$

- Piecewise-defined functions should be integrated separately, piece by piece,

$$f(x) = \begin{cases} f_1(x) & a \le x < x_1 \\ f_2(x) & x_1 \le x < x_2 \\ f_3(x) & x_2 \le x < b \end{cases}$$

$$\int_a^b f(x)\,dx = \int_a^{x_1} f_1(x)\,dx + \int_{x_1}^{x_2} f_2(x)\,dx + \int_{x_2}^{b} f_3(x)\,dx$$

- Integration by parts (with $g(x) = G'(x)$)

$$\int_a^b f(x)g(x)\,dx = f(x)G(x)\Big|_a^b - \int_a^b f'(x)G(x)\,dx.$$

- See review sheets for methods of integrations and trig identities...

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4 Convergence of infinite series

- A partial sum is the sum of the first N terms from the infinite series,

$$S_N(x) = \sum_{k=1}^{N} c_k \phi_k(x) \tag{7}$$

- Point-wise convergence: examine the series at a fixed point x_0, then the terms become constants, $a_k = c_k\phi_k(x_0)$

$$S_N(x_0) = \sum_{k=1}^{N} c_k\phi_k(x_0) = \sum_{k=1}^{N} a_k$$

  then convergence means that

$$\lim_{N\to\infty} S_N(x_0) = f(x_0)$$

  and questions of convergence one x-value-at-a-time are reduced to calculus questions about series of numbers:

  - n^{th} term test: a necessary condition for convergence is that

$$\lim_{k\to\infty} a_k = 0$$

    (i.e. if $a_{k\to\infty} \neq 0$ then the series does not converge). For alternating series (terms alternate in sign), if $|a_k|$ is decreasing as k increases and if this test is satisfied, then the series converges.
  - Important series

$$\text{Geometric series}: \quad \sum_{k=0}^{\infty} r^k = \frac{1}{1-r} \quad \text{converges if } |r| < 1 \qquad p\text{-series}: \quad \sum_{k=0}^{\infty} \frac{1}{k^p} \quad \text{converges if } p > 1$$

    Harmonic series: $\sum_k (1/k)$, diverges – it is a p-series with $p = 1$.
  - Limit comparison test: if $\sum_k b_k$ is known to converge or diverge, then if

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \text{finite nonzero value}$$

    then the $\sum_k a_k$ series will converge/diverge just like $\sum_k b_k$.
  - Absolute ratio test: examine the limit of the ratio of successive terms,

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| < 1: \quad \text{converges}, \qquad \lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| > 1: \quad \text{diverges}, \qquad \lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = 1: \quad \text{use a different test.}$$

  - If $\sum_k |a_k|$ converges, then the series is called absolutely convergent.

- Uniform convergence: $S_N(x)$ converges uniformly to $f(x)$ on $a \le x \le b$ if we can define the sequence $M_k = \max_x |S_k(x) - f(x)|$ and it is true that $M_k \to 0$ as $k \to \infty$. Uniform convergence is an important property: if your Fourier Series for $f(x)$ is uniformly convergent, then it is possible to guarantee when operations (like derivatives) on the series will give the correct (convergent) answer.