Math 211, Fall 2009, Duke University — Basic Mathematics Summary

Algebra: $ax^2 + bx + c = 0 \qquad x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Geometry

Triangle: the law of cosines

$$c^2 = a^2 + b^2 - 2ab\cos\phi \qquad \text{Area}=\tfrac{1}{2}(\text{base})(\text{height})$$

Parallelogram: Area=(base)(height)

Circle: center (h, k), radius r

$$(x-h)^2 + (y-k)^2 = r^2$$

$$\text{Area}=\pi r^2 \qquad \text{Circumference } s = 2\pi r$$

Ellipse: center (h, k), axes a, b

$$\left(\frac{x-h}{a}\right)^2 + \left(\frac{y-k}{b}\right)^2 = 1$$

Hyperbola: center (h, k), axes a, b

$$\left(\frac{x-h}{a}\right)^2 - \left(\frac{y-k}{b}\right)^2 = \pm 1$$

Parabola: tip (h, k)

$$y = a(x-h)^2 + k$$

Volumes

Cylinder	Volume=(base area)(height)	
Cone	Volume=$\frac{1}{3}$(base area)(height)	
Sphere	Volume=$(4/3)\pi r^3$	SurfaceArea $= 4\pi r^2$

Vectors, Matrices, and Linear Algebra

Multiplication: Row × column

$$\begin{pmatrix} \underline{a} & \underline{b} \\ c & d \end{pmatrix}\begin{pmatrix} \underline{x} & z \\ \underline{y} & w \end{pmatrix} = \begin{pmatrix} \underline{a}\underline{x} + \underline{b}\underline{y} & \underline{a}z + \underline{b}w \\ c\underline{x} + d\underline{y} & cz + dw \end{pmatrix}$$

Determinants

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Cramer's rule[1] to solve linear system $\mathbf{Ax} = \mathbf{f}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \underline{e} \\ \underline{f} \end{pmatrix}$$

$$x = \frac{\begin{vmatrix} \underline{e} & b \\ \underline{f} & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a & \underline{e} \\ c & \underline{f} \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

Inverse matrix: $\mathbf{Ax} = \mathbf{f} \quad \Rightarrow \quad \mathbf{x} = \mathbf{A}^{-1}\mathbf{f}$

$$\mathbf{A}^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Trig

Identities

$$\sin^2 x + \cos^2 x = 1 \qquad \sec^2 x = 1 + \tan^2 x$$

$$\tan x = \sin x/\cos x \qquad \sec x = 1/\cos x$$
$$\cot x = \cos x/\sin x \qquad \csc x = 1/\sin x$$

Complementary angles

$$\sin x = \cos(\pi/2 - x) \qquad \cos x = \sin(\pi/2 - x)$$
$$\tan x = \cot(\pi/2 - x) \qquad \cot x = \tan(\pi/2 - x)$$
$$\sec x = \csc(\pi/2 - x) \qquad \csc x = \sec(\pi/2 - x)$$

Addition formulas

$$\sin(x+y) = \sin x\cos y + \cos x\sin y$$
$$\cos(x+y) = \cos x\cos y - \sin x\sin y$$

Half-angle formulas (from $\cos(x + x)$)

$$\cos^2 x = \frac{1+\cos(2x)}{2} \qquad \sin^2 x = \frac{1-\cos(2x)}{2}$$

Product formulas (from ± Addition formulas)

$$\sin x\cos y = \tfrac{1}{2}(\sin(x+y) + \sin(x-y))$$

$$\sin(2x) = 2\sin x\cos x \qquad \cos(2x) = 2\cos^2 x - 1$$

Derivatives

$$(d/dx)\sin x = \cos x \qquad (d/dx)\cos x = -\sin x$$

$$(d/dx)\tan x = \sec^2 x \qquad (d/dx)\sec x = \sec x\tan x$$

$$(d/dx)\sin^{-1} x = \frac{1}{\sqrt{1-x^2}} \qquad (d/dx)\tan^{-1} x = \frac{1}{1+x^2}$$

Integrals $\displaystyle\int \sin^m x\cos^n x\,dx$

Via u-Substitution: if $m =$ odd, then let $u = \cos x$
if $n =$ odd, then let $u = \sin x$

$\sec x$ and $\tan x$ integrals[2]: SET, TOS, $\sec^2 x = 1 + \tan^2 x$

Polar coordinates

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$r = \sqrt{x^2 + y^2} \qquad \theta = \arctan(y/x)$$

Logs, Exponentials

Notations

$$\ln x = \log x = \log_e x$$

Properties

$$\ln(xy) = \ln x + \ln y \qquad e^{x+y} = e^x e^y$$

$$\ln(x^a) = a\ln x \qquad e^{ax} = (e^x)^a \qquad b^a = e^{a\ln b}$$
$$e^{\ln x} = \ln(e^x) = x$$

Integrals

$$\int \frac{du}{u} = \ln|u| + C \qquad \int e^{au}\,du = \frac{1}{a}e^{au} + C$$

Hyperbolic Trig

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \sinh x = \frac{e^x - e^{-x}}{2}$$

Identity

$$\cosh^2 x - \sinh^2 x = 1$$

Derivatives

$$(d/dx)\cosh x = \sinh x \qquad (d/dx)\sinh x = \cosh x$$

Integrals

$$\int \cosh x\,dx = \sinh x + C \qquad \int \sinh x\,dx = \cosh x + C$$

[1] The general formula for the solution is: n^{th} variable = (det with n^{th} column replaced by RHS) divided by (det of LHS matrix).
[2] u =Sec if Even power of Tan and u =Tan if Odd power of Sec.

Limits

Function $f(x)$ is continuous at $x = a$ if

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = \lim_{x\to a} f(x) = f(a)$$

If all the limits exist, then

$$\lim_{x\to a}[f(x) + g(x)] = \left[\lim_{x\to a} f(x)\right] + \left[\lim_{x\to a} g(x)\right]$$

$$\lim_{x\to a} f(x)g(x) = \left[\lim_{x\to a} f(x)\right]\left[\lim_{x\to a} g(x)\right]$$

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \left[\lim_{x\to a} f(x)\right] \Big/ \left[\lim_{x\to a} g(x)\right]$$

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right)$$

L'Hospital's rule: if $f(a)/g(a)$ "=" $0/0$ or ∞/∞, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Differential Calculus

Limit definition of derivative for $y = f(x)$

$$\frac{dy}{dx} = f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

Power rule: $y = x^r$

$$\frac{dy}{dx} = rx^{r-1}$$

Product rule: $y = f(x)g(x)$

$$\frac{dy}{dx} = f'(x)g(x) + f(x)g'(x)$$

Quotient rule: $y = f(x)/g(x)$

$$\frac{dy}{dx} = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

Chain rule: $y = f(g(x))$

$$\frac{dy}{dx} = f'(g(x))g'(x)$$

Taylor series for $f(t)$ at $t = t_0$

$$f(t) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\left. \frac{d^n f}{dt^n} \right|_{t=t_0} \right) (t - t_0)^n$$

Integral Calculus

Fundamental theorem of calculus: if $F'(x) = f(x)$ then

$$F(x) = \int_a^x f(t)\, dt$$

Anti-derivatives/Indefinite integrals

$$\int f(x)\, dx = F(x) + C$$

Definite integrals

$$\int_a^b f(x)\, dx = F(b) - F(a)$$

Riemann sum definition of definite integral

$$\int_a^b f(x)\, dx = \lim_{N\to\infty} \left(\sum_{n=0}^{N} f(x_n)\, \Delta x_n \right)$$

Integration by parts[3]

$$\int_a^b u(x)v'(x)\, dx = u(x)v(x)\Big|_a^b - \int_a^b u'(x)v(x)\, dx$$

Leibniz's rule

$$\frac{d}{dt}\left(\int_{a(t)}^{b(t)} f(x,t)\, dx \right) = f(b,t)\frac{db}{dt} - f(a,t)\frac{da}{dt} + \int_a^b \frac{\partial f}{\partial t}\, dx$$

A table of basic integrals

1) $\int u^n\, du = \frac{u^{n+1}}{n+1} + C \quad (n \neq -1)$

2) $\int \frac{du}{u} = \ln|u| + C$

3) $\int e^u\, du = e^u + C$

4) $\int \cos u\, du = \sin u + C$

5) $\int \sin u\, du = -\cos u + C$

6) $\int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C$

7) $\int \frac{du}{1+u^2} = \tan^{-1} u + C$

8) $\int \sec^2 u\, du = \tan u + C$

9) $\int \sec u \tan u\, du = \sec u + C$

10) $\int \sec u\, du = \ln|\sec u + \tan u| + C$

Brief review of methods of integration: u-substitutions, integration by parts, use of trigonometric identities, trigonometric substitutions, completing the square, partial fractions...

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[3] Also known as the integral form of the product rule $(uv)' = u'v + uv' \quad \to \quad uv' = (uv)' - u'v$.