A brief note to clear up some confusingly similar but different ways different books might present things about adjoint operators (and especially, "Are the adjoint eigenvalues λ or $\overline{\lambda}$?").

1 About the eigenvalues of the adjoint operator

For a given linear operator matrix $\mathbf{L}$, the equation defining the eigenvalues and eigenvectors is just

$$\mathbf{L}\boldsymbol{\phi} = \lambda\boldsymbol{\phi} \tag{1}$$

So, the eigenvalues and eigenvectors are "set in stone" and they DO NOT care which inner product we will use to define orthogonality.

Lets compare what happens with the two standard choices for inner products, the real and complex dots products of vectors:

$$\langle \mathbf{x}, \mathbf{y}\rangle_R \equiv \mathbf{x}\cdot\mathbf{y} \qquad \langle \mathbf{x}, \mathbf{y}\rangle_C \equiv \mathbf{x}\cdot\overline{\mathbf{y}} \tag{2}$$

It is a bad idea to use the real dot product with complex vectors because it does not have the "norm property", but it can still be done if you'd really like to (and some textbooks use it), as long as you stay consistent and use only one version of the inner product!

So ignoring whether things are real or complex, we can try using both inner products and see what they produce. Recall that the adjoint operator is defined by inner product relation:

$$\langle \mathbf{y}, \mathbf{L}\mathbf{x}\rangle = \langle \mathbf{L}^*\mathbf{y}, \mathbf{x}\rangle \tag{3}$$

Which inner product you use changes what the adjoint operator $\mathbf{L}^*$ is!

$$\text{real inner product} \qquad \langle \mathbf{x}, \mathbf{y}\rangle_R \qquad \to \qquad \mathbf{L}^*_R = \mathbf{L}^T \tag{4}$$

$$\text{complex inner product} \qquad \langle \mathbf{x}, \mathbf{y}\rangle_C \qquad \to \qquad \mathbf{L}^*_C = \overline{\mathbf{L}^T} = \mathbf{L}^H \tag{5}$$

In problems where $\mathbf{L}^*_R \neq \mathbf{L}^*_C$ it WILL make a difference in the eigenvalues and eigenvectors for the adjoint.
Here are the general results for the adjoint eigenvalues λ^* in terms of the eigenvalues of the original operator $\mathbf{L}$:

$$\text{real inner product} \qquad \langle \mathbf{x}, \mathbf{y}\rangle_R \qquad \to \qquad \mathbf{L}^*_R = \mathbf{L}^T \qquad \to \qquad \boxed{\lambda^*_R = \lambda} \tag{6}$$

$$\text{complex inner product} \qquad \langle \mathbf{x}, \mathbf{y}\rangle_C \qquad \to \qquad \mathbf{L}^*_C = \overline{\mathbf{L}^T} \qquad \to \qquad \boxed{\lambda^*_C = \overline{\lambda}} \tag{7}$$

1.1 Matrix Example

Here is an example

$$\mathbf{L} = \begin{pmatrix} i & 2i \\ 2 & -1 \end{pmatrix} \qquad \begin{matrix} \lambda_1 = 1+2i & \boldsymbol{\phi}_1 = (1+i, 1)^T \\ \lambda_2 = -2-i & \boldsymbol{\phi}_2 = (-1-i, 2)^T \end{matrix} \tag{8}$$

Using the real inner product we get:

$$\mathbf{L}^*_R = \begin{pmatrix} i & 2 \\ 2i & -1 \end{pmatrix} \qquad \begin{matrix} \lambda^*_{R1} = 1+2i & \boldsymbol{\psi}_1 = (2, 1+i)^T \\ \lambda^*_{R2} = -2-i & \boldsymbol{\psi}_2 = (1, -1-i)^T \end{matrix} \tag{9}$$

and it is easy to check the orthogonality $\langle \boldsymbol{\phi}, \boldsymbol{\psi}\rangle_R$. Using the complex inner product we get:

$$\mathbf{L}^*_C = \begin{pmatrix} -i & 2 \\ -2i & -1 \end{pmatrix} \qquad \begin{matrix} \lambda^*_{C1} = 1-2i & \boldsymbol{\psi}_1 = (2, 1-i)^T \\ \lambda^*_{C2} = -2+i & \boldsymbol{\psi}_2 = (1, -1+i)^T \end{matrix} \tag{10}$$

and it is easy to check the orthogonality $\langle \boldsymbol{\phi}, \boldsymbol{\psi}\rangle_C$. Notice that these adjoint eigenvalues are the conjugates of the original ones due to the use of the complex inner product, where in the real inner product case, we got $\lambda^* = \lambda$.

> <u>The moral of this note</u>: Be careful about which inner-product you are using! Using the complex inner product for all of your calculations is always safe (but if everything is real-valued, it will just reduce back to the real-result anyway).