Math 211, Fall 2009, Duke University

Notes for Lecture 32: Complex integrals and the Complex Fourier Series

The Fourier series for a periodic function $f(\theta)$ on $0 \le \theta < 2\pi$:

$$f(\theta) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(n\theta) + \sum_{n=1}^{\infty} b_n \sin(n\theta)$$

where

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos(n\theta)\,d\theta \qquad b_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin(n\theta)\,d\theta$$

We will now see another way to derive the formulas for these coefficients.
Re-write sin, cos in terms of complex exponentials

$$\begin{aligned}
f(\theta) &= \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\left(\frac{e^{in\theta}+e^{-in\theta}}{2}\right) + \sum_{n=1}^{\infty} b_n\left(\frac{e^{in\theta}-e^{-in\theta}}{2i}\right)\\
&= \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(\frac{a_n - ib_n}{2}\right)e^{in\theta} + \sum_{n=1}^{\infty}\left(\frac{a_n+ib_n}{2}\right)e^{-in\theta}\\
&= c_0 + \sum_{n=1}^{\infty} c_n e^{in\theta} + \sum_{n=1}^{\infty} c_{-n}e^{-in\theta}
\end{aligned}$$

where

$$c_n = \frac{a_n - ib_n}{2} \qquad c_{-n} = \frac{a_n + ib_n}{2} \qquad c_0 = \frac{a_0}{2} \qquad b_0 = 0$$

Compact final form:

$$\boxed{f(\theta) = \sum_{n=-\infty}^{\infty} c_n e^{in\theta}} \qquad \boxed{\text{The Complex Fourier series}}$$

Define $f(\theta) = F(e^{i\theta})$, then having $z = e^{i\theta}$ as the points on the circle $|z| = 1$, extend $F(z)$ to the rest of the complex plane as

$$F(z) = \sum_{n=-\infty}^{\infty} c_n z^n \tag{1}$$

If $F(z)$ is an analytic function then $c_n = 0$ for all $n < 0$ and the series reduces to a Taylor series,

$$F(z) = \sum_{n=0}^{\infty} c_n z^n \qquad c_n = \frac{F^{(n)}(0)}{n!} \tag{2}$$

We can rewrite the derivatives in the Taylor series coefficients using the Cauchy Integral formula for $F^{(n)}(0)$,

$$c_n = \frac{1}{n!}\left[\frac{n!}{2\pi i}\oint_C \frac{F(z)}{z^{n+1}}\,dz\right] \tag{3}$$

and finally, evaluate the contour integrals on $|z| = 1$ using the parametrization $z(\theta) = e^{i\theta}$:

$$\begin{aligned}
c_n = \frac{1}{2\pi i}\int_0^{2\pi}\frac{F(e^{i\theta})ie^{i\theta}}{e^{i(n+1)\theta}}\,d\theta &= \boxed{c_n = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)e^{-in\theta}\,d\theta}\\
&= \underbrace{\frac{1}{2\pi}\int_0^{2\pi} f(\theta)\cos(n\theta)\,d\theta}_{\frac{1}{2}a_n} \underbrace{-\frac{i}{2\pi}\int_0^{2\pi} f(\theta)\sin(n\theta)\,d\theta}_{-\frac{i}{2}b_n}
\end{aligned}$$