Math 211, Fall 2009, Duke University — Review of Infinite Series

A quick review of some important infinite series

Many, many, many infinite series can be derived in one or two steps[1] from the three basic series; (1) the exponential,a (2) the geometric series, and (3) the binomial expansion. The key idea is that whatever you can do to the function $S(w)$ representing the convergent series, you can do, term by term, to each term in the series; differentiation, integration, substitution, factoring, etc. As an example of substitution, the variable w below stands for $w = \textbf{Whatever}(z)$, so if $w = z^4$ then we get the series for $\tan^{-1}(z^4) = z^4 - z^{12}/3 + \cdots$.

1) The exponential series

$$\mathbf{e^w} = \mathbf{1} + \mathbf{w} + \frac{\mathbf{w^2}}{\mathbf{2!}} + \frac{\mathbf{w^3}}{\mathbf{3!}} + \frac{\mathbf{w^4}}{\mathbf{4!}} + \frac{\mathbf{w^5}}{\mathbf{5!}} + \frac{\mathbf{w^n}}{\mathbf{n!}} + \cdots$$

$$\frac{1}{2}(e^{iw} + e^{-iw}) = \cos(w) = 1 - \frac{w^2}{2!} + \frac{w^4}{4!} + \frac{(-1)^n w^{2n}}{(2n)!} + \cdots$$

$$\frac{1}{2i}(e^{iw} - e^{-iw}) = \sin(w) = w - \frac{w^3}{3!} + \frac{w^5}{5!} + \frac{(-1)^n w^{2n+1}}{(2n+1)!} + \cdots$$

$$\frac{1}{2}(e^{w} + e^{-w}) = \cosh(w) = 1 + \frac{w^2}{2!} + \frac{w^4}{4!} + \frac{w^{2n}}{(2n)!} + \cdots$$

$$\frac{1}{2}(e^{w} - e^{-w}) = \sinh(w) = w + \frac{w^3}{3!} + \frac{w^5}{5!} + \frac{w^{2n+1}}{(2n+1)!} + \cdots$$

2) The geometric series[2] $\boxed{|\mathbf{w}| < \mathbf{1}}$

$$\frac{\mathbf{1}}{\mathbf{1-w}} = \mathbf{1} + \mathbf{w} + \mathbf{w^2} + \mathbf{w^3} + \cdots + \mathbf{w^n} + \cdots$$

$$\frac{d}{dw}\left(\frac{1}{1-w}\right) = \frac{1}{(1-w)^2} = 1 + 2w + 3w^2 + \cdots + nw^{n-1} \cdots$$

$$\int \frac{dw}{1-w} = -\ln(1-w) = w + \frac{w^2}{2} + \frac{w^3}{3} + \frac{w^4}{4} + \cdots + \frac{w^n}{n} + \cdots$$

$$\int \frac{dw}{1+w^2} = \tan^{-1}(w) = w - \frac{w^3}{3} + \frac{w^5}{5} - \frac{w^7}{7} + \cdots + \frac{(-1)^n w^{2n+1}}{2n+1}$$

3) The binomial series $\boxed{|\mathbf{w}| < \mathbf{1}, \text{ any } \mathbf{r}}$

$$(\mathbf{1} + \mathbf{w})^{\mathbf{r}} = \mathbf{1} + \mathbf{rw} + \frac{\mathbf{r(r-1)w^2}}{\mathbf{2!}} + \frac{\mathbf{r(r-1)(r-2)w^3}}{\mathbf{3!}} + \cdots$$

$$(1-w)^{-1} = 1 + w + w^2 + w^3 + \cdots$$

$$\sqrt{1+w^2} = 1 + \frac{w^2}{2} - \frac{w^4}{8} + \frac{w^6}{16} - \cdots$$

$$\int \frac{dw}{\sqrt{1-w^2}} = \sin^{-1}(w) = w + \frac{w^3}{6} + \frac{3w^5}{40} + \frac{5w^7}{112} + \cdots$$

- Power Series at z_0: $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ Taylor Series: coefficients $a_n = \dfrac{f^{(n)}(z_0)}{n!}$
  A Taylor series for $z_0 = 0$ is also sometimes called a MacLaurin series.
  For some functions, obtaining the infinite series by relating it to one of the three families above can be a much quicker/easier approach than working out the Taylor series coefficients.
- Laurent Series at z_0: $f(z) = \sum_{n=1}^{\infty} \dfrac{b_n}{(z - z_0)^n} + \sum_{n=0}^{\infty} a_n (z - z_0)^n$
  The residue of $f(z)$ at z_0 is b_1. Taylor series are Laurent series with all $b_n = 0$.

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[1] Using operations like algebra, composition, integration, differentiation...

[2] The formula for the sum of a finite part of a geometric series is "(first term - ratio times the last term)/(1-ratio)"