Math 211, Fall 2009, Duke University — Lecture 30

Introduction to complex contour integrals

Review of line integrals from vector calculus: Work integrals in 2-D

Work from physics/mechanics = force applied push an object along a path

$$\text{Work} = \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{x}}$$

Force = fcn(position): $\vec{\mathbf{F}}(\vec{\mathbf{x}}) = P(x,y)\hat{\mathbf{i}} + Q(x,y)\hat{\mathbf{j}}$

Curve C: $\vec{\mathbf{x}}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}}$ Parametric equations for path curve

starting point at $t = a$: $\vec{\mathbf{x}}(a) = \vec{\mathbf{A}}$ $\longrightarrow$ ending point at $t = b$: $\vec{\mathbf{x}}(b) = \vec{\mathbf{B}}$

1. **Direct evaluation**: using parametric equations for C

$$\begin{aligned}
\text{Work} &= \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{x}} = \int_C P\,dx + Q\,dy \\
&= \int_a^b \vec{\mathbf{F}}(\vec{\mathbf{x}}(t)) \cdot \left(\frac{d\vec{\mathbf{x}}}{dt}\,dt\right) = \int_a^b \left[\vec{\mathbf{F}}(\vec{\mathbf{x}}(t)) \cdot \frac{d\vec{\mathbf{x}}}{dt}\right] dt \\
&= \int_a^b \left[P(x(t),y(t))\frac{dx}{dt} + Q(x(t),y(t))\frac{dy}{dt}\right] dt \\
&= \int_a^b M(t)\,dt
\end{aligned}$$

Example via parametrization

$$I = \int_C e^y\,dx + \sin(xy)\,dy = \int_{C_1} + \int_{C_2}$$

$$C_1: \qquad x(t) = 2\cos t \qquad y(t) = 2\sin t \qquad t: 0 \to \pi/4$$

$$C_2: \qquad x(t) = t \qquad y(t) = t \qquad t: \sqrt{2} \to 0$$

$$I_1 = \int_0^{\pi/4} \left[e^{2\sin t}(-2\sin t) + \sin(4\cos t \sin t)(2\cos t)\right]\,dt$$

$$I_2 = \int_{\sqrt{2}}^{0} \left[e^t \cdot 1 + \sin(t^2)\cdot 1\right]\,dt \qquad I = I_1 + I_2$$

General property for all line integrals: work for reverse trip on the same curve

$$\int_{-C} \vec{\mathbf{F}} \cdot d\vec{\mathbf{x}} = \int_b^a \left[\vec{\mathbf{F}}(\vec{\mathbf{x}}(t)) \cdot \frac{d\vec{\mathbf{x}}}{dt}\right]\,dt = -\int_a^b \cdots\,dt = -\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{x}}$$

2. **Path-independent integrals**: can be computed just from endpts!

For some $\vec{\mathbf{F}}(\vec{\mathbf{x}})$ fcns, ANY path $C_1, C_2, \cdots$ from $\vec{\mathbf{A}}$ to $\vec{\mathbf{B}}$ will yield the same work

$$\left[\int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{x}} = \int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{x}} = \cdots\right] = \int_{\vec{\mathbf{A}}}^{\vec{\mathbf{B}}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{x}}$$

Q: What for $\vec{\mathbf{F}}$'s is this true?

A: If $\vec{\mathbf{F}}$ doesn't blow-up and is smooth and

$\boxed{\vec{\mathbf{F}} \text{ is the gradient of some fcn: } \vec{\mathbf{F}} = \nabla\phi(\vec{\mathbf{x}})}$

Then $\vec{\mathbf{F}}$ is called a conservative vector field

Q: How do we test if $\vec{\mathbf{F}} = \nabla\phi$?

A: A vector identity: $\nabla \times (\nabla\phi) \equiv \vec{\mathbf{0}}$ for all $\phi(\vec{\mathbf{x}})$ fcns:

If $\boxed{\nabla \times \vec{\mathbf{F}} \equiv \vec{\mathbf{0}}}$ then work is path-independent: $\displaystyle\int_{\vec{\mathbf{A}}}^{\vec{\mathbf{B}}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{x}} = \phi(\vec{\mathbf{B}}) - \phi(\vec{\mathbf{A}})$

In 2D: $\vec{\mathbf{F}}(\vec{\mathbf{x}}) = P(x,y)\hat{\mathbf{i}} + Q(x,y)\hat{\mathbf{j}} \quad \longrightarrow \quad \nabla \times \vec{\mathbf{F}} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{\mathbf{k}}$

Path independent in 2D if $\boxed{\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}}$

3. <u>**Green's Theorem in 2D**</u>

$$\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{x}} = \iint_D (\nabla \times \vec{\mathbf{F}}) \cdot \hat{\mathrm{k}}\, dA$$

$$\boxed{\oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA}$$

where D is the region inside the closed curve C

- Green's Theorem holds as long as $\vec{\mathbf{F}}$ and its derivatives do not blow-up inside C (else no promises, $\oint \neq \iint$)
- Consider Green's theorem applied to a conservative $\vec{\mathbf{F}}(\vec{\mathbf{x}})$:
  - LHS using path independence for a "round-trip" between $\vec{\mathbf{A}}, \vec{\mathbf{B}}$:

$$\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{x}} = \int_{\vec{\mathbf{A}}}^{\vec{\mathbf{B}}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{x}} + \int_{\vec{\mathbf{B}}}^{\vec{\mathbf{A}}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{x}} = 0$$

  - RHS is confirmed from the curl condition, $\nabla \times \vec{\mathbf{F}} = \vec{\mathbf{0}}$:

$$\iint_D (\nabla \times \vec{\mathbf{F}}) \cdot \hat{\mathrm{k}}\, dA = \iint_D 0\, dA = 0$$