Math 211, Fall 2008, Duke University

How to Evaluate Multi-valued Functions with Branch Cuts

To evaluate functions with roots (for example, a function with two factors, $A > 0$)

$$f(z) = A(z - z_1)^{1/n}(z - z_2)^{1/m}$$

Follow these steps:

1. Write the function in exponential form

$$f(z) = e^{\ln f(z)} \cdot 1 = e^{\ln A + \frac{1}{n}\ln(z-z_1) + \frac{1}{m}\ln(z-z_2) + i\,2\pi k_0}$$

2. Branch points are given by the z values where any of the log terms blow up ($\ln 0$ or $\ln \infty$). Each log term can be written in its own polar form

$$\ln(z - z_1) = \ln|z - z_1| + i\arg(z - z_1) = \ln r_1 + i(\theta_1 + 2\pi k_1)$$

$$\ln(z - z_2) = \ln|z - z_2| + i\arg(z - z_2) = \ln r_2 + i(\theta_2 + 2\pi k_2)$$

That is, r_1 is the distance from the center z_1 to the point z and θ_1 is the angle between z and the positive horizontal axis starting from the point z_1, similarly for r_2, θ_2 for the distance and angle of z with respect to center point z_2.
Using this, you can write $f(z)$ as

$$f(z) = A r_1^{1/n} r_2^{1/m} \exp\left[i\left(\frac{\theta_1 + 2\pi k_1}{n} + \frac{\theta_2 + 2\pi k_2}{m}\right)\right]$$

where the $\arg(f(z))$ can be called the total phase.

3. Pick branch cuts that connect the branch points to each other, or each branch point to infinity. This defines a range of angles on each Riemann sheet, but does not yet select which sheet you are on (which k in $2\pi k$, one for each branch point). There is one angle associated with each branch point. Some typical choices for branch cuts are

   - Negative real axis, $\big(-\pi < \theta \le \pi\big) + 2\pi k$
   - Positive real axis, $\big(0 \le \theta < 2\pi\big) + 2\pi k$
   - Negative imaginary axis, $\big(-\pi/2 < \theta \le 3\pi/2\big) + 2\pi k$
   - Positive imaginary axis, $\big(-3\pi/2 < \theta \le \pi/2\big) + 2\pi k$

   The total phase of $f(z)$ jumps in value when you cross a branch cut, and changes smoothly everywhere else – you can use this to check your choices of angles.

4. To decide which Riemann sheet you are on; someone must either tell you what k_1 and k_2 are, or give you the value of $f(z)$ at some arbitrary point (but it shouldn't be on a branch cut) – then you must pick choices for k_1 and k_2 will give you that same value of $f(z)$ at the point z. Example: $f(z) = \sqrt{z}$, given that $f(4) = -2$, you realize that the formula for $f(z)$ on this Riemann sheet is really $f(z) = \sqrt{r}\exp[i(\theta + 2\pi)/2]$.

5. Once you have determined k_1 and k_2, you can determine the value of $f(z)$ anywhere on the Riemann sheet. To answer the question $f(4 + 3i) =?$, just figure out the distances and angles to $z = 4 + 3i$ and plug them into the polar form for $f(z)$.