Math 211, Fall 2009, Duke University Lecture 28

Introduction to functions of complex variables: motivation

- An extension of real numbers to a complete set: $\sqrt{-3}$ is not real, but every basic operation on every $\mathbb{C}$ number yields a $\mathbb{C}$ answer
- An extension of usual real functions that makes some calculations (many integrals and differential equations) easier

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- The unit imaginary number $\boxed{\mathbf{i} = \sqrt{-1}}$, with $\mathbf{i}^2 = -1$
- If x, y are real numbers ($\mathbb{R}$) then $\boxed{z = x + \mathbf{i}y}$ is complex ($\mathbb{C}$)
- $z \in \mathbb{C}$ have some "2D vector-ish" properties
  - Separable components: Real/Imaginary parts $\mathbf{Re}(z) = x$ and $\mathbf{Im}(z) = y$
  - Componentwise addition: $z_1 + z_2 = (x_1 + x_2) + \mathbf{i}(y_1 + y_2)$
- But products are done as usual algebra (not "vectors")

$$(a + \mathbf{i}b)(c + \mathbf{i}d) = ac + \mathbf{i}ad + \mathbf{i}bc + \mathbf{i}^2 bd = (ac - bd) + \mathbf{i}(ad + bc)$$

- Conjugation: negation of imaginary part: $\boxed{\overline{\mathbf{i}} = -\mathbf{i}}$ Example: $\overline{3 + \mathbf{i}4} = 3 - \mathbf{i}4$

- Complex conjugate of $z = x + \mathrm{i}y$ is $\overline{z} = x - \mathrm{i}y$
- Formulas for components of z:

$$\mathrm{Re}(z) \equiv \frac{z + \overline{z}}{2} = x \qquad \mathrm{Im}(z) \equiv \frac{z - \overline{z}}{2\mathrm{i}} = y$$

- Modulus (magnitude or length) of z

$$|z|^2 \equiv z\overline{z} = (x + \mathrm{i}y)(x - \mathrm{i}y) = x^2 + y^2$$

- Every real function $f(x)$ can be "complexified" by evaluating it at $f(z) = f(x + \mathrm{i}y)$. Fcn $f(z)$ yields complex values. Examples:
  - $f(x) = x^2 \quad \longrightarrow \quad \boxed{f(z) = z^2}$
    $f(z) = (x + \mathrm{i}y)^2 = (x^2 - y^2) + \mathrm{i}(2xy)$
  - $f(x) = e^x \quad \longrightarrow \quad \boxed{f(z) = e^z}$
    $f(z) = e^{x+iy} = e^x e^{iy} = e^x\,(\cos y + \mathrm{i}\sin y)$ (via Euler's formula)
    $f(z) = (e^x \cos y) + \mathrm{i}(e^x \sin y)$

$\boxed{f(z) = u(x,y) + \mathrm{i}v(x,y)}$ u, v: real "component" functions of f

$$u(x,y) = \mathrm{Re}(f) \qquad \text{and} \qquad v(x,y) = \mathrm{Im}(f)$$

- $z = x + \mathrm{i}y$ is the rectangular coordinate form
- $\boxed{z = re^{i\theta}}$ is the polar coordinate form $(r, \theta \in \mathbb{R})$

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$z = x + \mathrm{i}y = r(\cos\theta + \mathrm{i}\sin\theta) = re^{\mathrm{i}\theta}$$

- Rectangular to Polar conversion

$$r = |z| = \sqrt{x^2 + y^2} \qquad \theta = \arg(z) = \tan^{-1}(y/x)$$

- Multiplication is easier in polar form

$$z_1 z_2 = (r_1 e^{\mathrm{i}\theta_1})(r_2 e^{\mathrm{i}\theta_2}) = (r_1 r_2)e^{\mathrm{i}(\theta_1+\theta_2)}$$

- Euler's formula: $\boxed{e^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta}$ So $\boxed{e^{\mathrm{i}2\pi k} = 1 \text{ for any } k = \text{integer}}$

  Identity for $k = 0:$ $\qquad z = z{\cdot}1 \qquad \longrightarrow \qquad f(z) = f(z{\cdot}1)$ $\qquad$ <u>Always true</u>

  But Amazingly, there are functions where for other integer-values of k:

$$f(z) \neq f(ze^{\mathrm{i}2\pi k}) \qquad \underline{\text{Multi-valued functions}}$$

Examples of Multi-valued complex fcns: $z = ze^{\mathrm{i}2\pi k}$ but $f(z) \neq f(ze^{\mathrm{i}2\pi k})$

1. The square root function $f(z) = \sqrt{z}$
   Use polar coordinates $z = re^{\mathrm{i}\theta}$:

$$\begin{aligned}
f(z) &= \left(re^{\mathrm{i}\theta}\right)^{1/2} = r^{1/2}e^{\mathrm{i}\theta/2} \\
f(ze^{\mathrm{i}2\pi k}) &= \left(re^{\mathrm{i}[\theta+2\pi k]}\right)^{1/2} = r^{1/2}e^{\mathrm{i}[\theta+2\pi k]/2} \\
&= r^{1/2}e^{\mathrm{i}\theta/2}e^{\mathrm{i}\pi k} = \begin{cases} +r^{1/2}e^{\mathrm{i}\theta/2} & k \text{ is even, } e^{\mathrm{i}\pi k} = 1 \\ -r^{1/2}e^{\mathrm{i}\theta/2} & k \text{ is odd, } e^{\mathrm{i}\pi k} = -1 \end{cases} \\
f(z) &= \pm z^{1/2} \qquad \text{Multi-valued} \longrightarrow \text{2 values or ``branches''}
\end{aligned}$$

2. The logarithm $f(z) = \ln(z)$

$$\begin{aligned}
f(z) &= \ln(re^{\mathrm{i}\theta}) = \ln(r) + \ln(e^{\mathrm{i}\theta}) = \ln(r) + \mathrm{i}\theta \\
f(ze^{\mathrm{i}2\pi k}) &= \ln(re^{\mathrm{i}[\theta+2\pi k]}) \\
&= \ln(r) + \ln(e^{\mathrm{i}[\theta+2\pi k]}) \\
&= \ln(r) + \mathrm{i}[\theta + 2\pi k] \\
&= \ln(z) + i2\pi k \qquad \text{infinite branches}
\end{aligned}$$