1 The Laplacian in different coordinate systems

- Rectangular coordinates (2D) $u = u(x, y)$ or (3D) $u = u(x, y, z)$

$$\boxed{\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}} \qquad \boxed{\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}}$$

- Cylindrical polar coordinates (2D) $u = u(r, \theta)$ or (3D) $u = u(r, \theta, z)$

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

$$r = \sqrt{x^2 + y^2} \qquad \theta = \arctan(y/x) \qquad z = z$$

$$\boxed{\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}} \qquad \boxed{\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}}$$

- Spherical polar coordinates $u = u(\rho, \theta, \phi)$

$$x = \rho\sin\phi\cos\theta \qquad y = \rho\sin\phi\sin\theta \qquad z = \rho\cos\phi$$

$$\rho = \sqrt{x^2 + y^2 + z^2} \qquad \theta = \arctan(y/x) \qquad \phi = \arctan(r/z)$$

$$\boxed{\nabla^2 u = \frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2\frac{\partial u}{\partial \rho}\right) + \frac{1}{\rho^2\sin^2\phi}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{\rho^2\sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi\frac{\partial u}{\partial \phi}\right)}$$

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2 ODE BVPs from Separation-of-Variables for the Helmholtz eqn $\nabla^2\phi = -\lambda\phi$

1) General self-adjoint eigenvalue problems
Sturm-Liouville equation on $a \le x \le b$

$$Ly = -\lambda\sigma(x)y$$

$$\boxed{\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y = -\lambda\sigma(x)y}$$

Singular problem:
If $p(a) = 0$, then no BC at $x = a$ (similar at $x = b$)
General solution (homogeneous solution)

$$\boxed{y_h(x) = d_1 w_1(x) + d_2 w_2(x)}$$

Homogeneous BC's select the eigensolns $(\lambda_n,\ y_n(x))$ for $n = 0, 1, 2, ...$ from the general solution.

Inner product with weight function $\sigma(x)$

$$\langle y_n, y_m\rangle = \int_a^b y_n(x)y_m(x)\sigma(x)\,dx$$

Orthogonality $\langle y_n, y_m\rangle = 0$ if $n \neq m$
Norm $||y_n||^2 = \langle y_n, y_n\rangle$
Series expansion, coefficients

$$f(x) = \sum_{n=1}^{\infty} c_n y_n(x), \qquad c_n = \frac{\langle f(x), y_n(x)\rangle}{||y_n||^2}$$

2) Rectangular coordinates: $f(x), g(y)$ or $g(\theta)$
Harmonic oscillator equation on $0 \le x \le \ell$

$$\boxed{y'' = -\lambda y}$$

SL coefficient fcns: $p(x) = 1$, $q(x) = 0$, $\sigma(x) = 1$
General solution $\lambda > 0$

$$\boxed{y_h(x) = d_1 \cos(\sqrt{\lambda}x) + d_2 \sin(\sqrt{\lambda}x)}$$

Inner product with weight function $\sigma(x) = 1$

$$\langle y_n, y_m \rangle = \int_0^\ell y_n(x) y_m(x)\, dx$$

Useful formulas:

$$\sin(n\pi) = 0, \qquad \cos(n\pi) = (-1)^n$$

$$\sin\left(\tfrac{(2n+1)\pi}{2}\right) = (-1)^n, \qquad \cos\left(\tfrac{(2n+1)\pi}{2}\right) = 0$$

$$||\sin(\sqrt{\lambda}x)||^2 = ||\cos(\sqrt{\lambda}x)||^2 = \ell/2, \qquad \lambda > 0$$

$$||1||^2 = \ell, \qquad \big(\cos(\sqrt{\lambda}x) \equiv 1, \quad \lambda = 0\big)$$

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3) Polar coordinates (Laplace): $f(r)$
Cauchy-Euler equation on $a \le x \le b$

$$\boxed{x^2 y'' + x y' = -\lambda y}$$

SL coefficient fcns: $p(x) = x$, $q(x) = 0$, $\sigma(x) = 1/x$
Singular at $x = 0$: $p(0) = 0$
General solution $\lambda > 0$, $(m = \pm i\sqrt{\lambda})$

$$\boxed{y_h(x) = d_1 \cos\big(\sqrt{\lambda}\ln(x)\big) + d_2 \sin\big(\sqrt{\lambda}\ln(x)\big)}$$

Inner product with weight function $\sigma(x) = 1/x$

$$\langle y_n, y_m \rangle = \int_a^b y_n(x) y_m(x) \frac{1}{x}\, dx$$

Useful formulas:

$$(xy')' = -\lambda x^{-1} y$$

$$||\sin\big(\sqrt{\lambda}\ln(x)\big)||^2 = ||\cos\big(\sqrt{\lambda}\ln(x)\big)||^2 = \frac{1}{2}\ln(b/a)$$

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4) Polar coordinates (Helmholtz): $f(r)$
Bessel's equation of order k on $a \le x \le b$

$$\boxed{x^2 y'' + x y' - k^2 y = -\lambda x^2 y}$$

SL coefficient fcns: $p(x) = x$, $q(x) = -k^2/x$, $\sigma(x) = x$
Singular at $x = 0$: $p(0) = 0$
General solution $\lambda > 0$

$$\boxed{y_h(x) = d_1 \mathrm{J}_k(\sqrt{\lambda}x) + d_2 \mathrm{Y}_k(\sqrt{\lambda}x)}$$

Inner product with weight function $\sigma(x) = x$

$$\langle y_n, y_m \rangle = \int_a^b y_n(x) y_m(x) x\, dx$$

Useful formulas:

$$(xy')' - (k^2/x) y = -\lambda x y$$

$$\mathrm{J}_0(0) = 1, \qquad \mathrm{J}_0'(0) = 0$$

$$\mathrm{J}_k(0) = 0, \qquad \mathrm{Y}_k(0) = -\infty, \qquad k = 1, 2, 3...$$

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5) Spherical coordinates: $g(\phi)$
Legendre's equation on $-1 \le x \le 1$ or $0 \le x \le 1$

$$\boxed{\big((1 - x^2) y'\big)' = -\lambda y}$$

SL coefficient fcns: $p(x) = 1 - x^2$, $q(x) = 0$, $\sigma(x) = 1$
Singular at $x = \pm 1$: $p(\pm 1) = 0$
General solution $\lambda > 0$

$$\boxed{y_h(x) = d_1 \mathrm{P}_\lambda(x) + d_2 \mathrm{Q}_\lambda(x)}$$

Inner product with weight function $\sigma(x) = 1$

$$\langle y_n, y_m \rangle = \int_{x_1}^{x_2} y_n(x) y_m(x)\, dx$$

Useful formulas:

$$\boxed{g(\phi) = y(\cos\phi), \qquad x = \cos\phi}$$

$$\boxed{(\sin\phi\, g')' + \lambda \sin\phi\, g = 0}$$

$$\mathrm{P}_\lambda(x) = \text{“polynomial”}, \qquad \mathrm{Q}_\lambda(\pm 1) = \infty$$

For $-1 \le x \le 1$ (no BC's needed)

$$\lambda_n = n(n+1), \qquad y_n(x) = P_n(x), \qquad n = 0, 1, 2, ...$$

For $0 \le x \le 1$

$$y'(0) = 0, \qquad \lambda_n = 2n(2n+1), \qquad y_n(x) = P_{2n}(x)$$

$$y(0) = 0, \qquad \lambda_n = 2(n+1)(2n+1), \qquad y_n(x) = P_{2n+1}(x)$$

	$\lambda > 0$ Oscillatory solutions	$\lambda < 0$ Non-oscillatory solns
Harmonic, $f(x)$, $g(\theta)$	$\sin(\sqrt{\lambda}x), \cos(\sqrt{\lambda}x)$	$\sinh(\alpha x), \cosh(\alpha x)$ or $e^{\pm\alpha x}$
Cauchy-Euler, $f(r)$	$\sin(\sqrt{\lambda}\ln r), \cos(\sqrt{\lambda}\ln r)$	$r^\alpha, r^{-\alpha}$
Bessel order k, $f(r)$	$\mathrm{J}_k(\sqrt{\lambda}r), \mathrm{Y}_k(\sqrt{\lambda}r)$	$\mathrm{I}_k(\sqrt{\lambda}r), \mathrm{K}_k(\sqrt{\lambda}r)$
Legendre, $g(\phi)$	$\mathrm{P}_n(\cos\phi), \mathrm{Q}_n(\cos\phi)$	—