

Basics of Bessel functions

The Bessel functions are solutions of versions of Bessel's equation. Bessel equations are produced by separation of variables from the Helmholtz equation, $\nabla^2 \phi = -\lambda \phi$, in polar coordinates. See Haberman, Chapter 7.7.

1 Bessel's equation of order zero

The Bessel equation for order zero comes out of separation of variables for problems that have no dependence on θ :

$$\boxed{\frac{d}{dz} \left(z \frac{dy}{dz} \right) + zy = 0} \quad (1)$$

General solution:

$$\boxed{y(z) = c_1 J_0(z) + c_2 Y_0(z)} \quad (2)$$

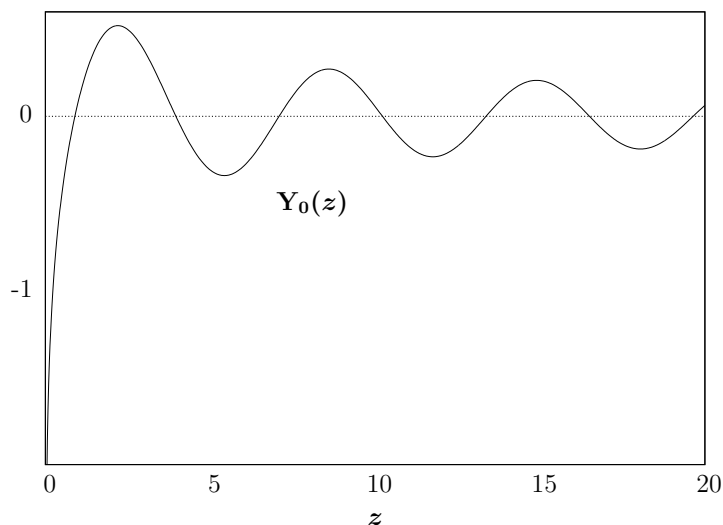
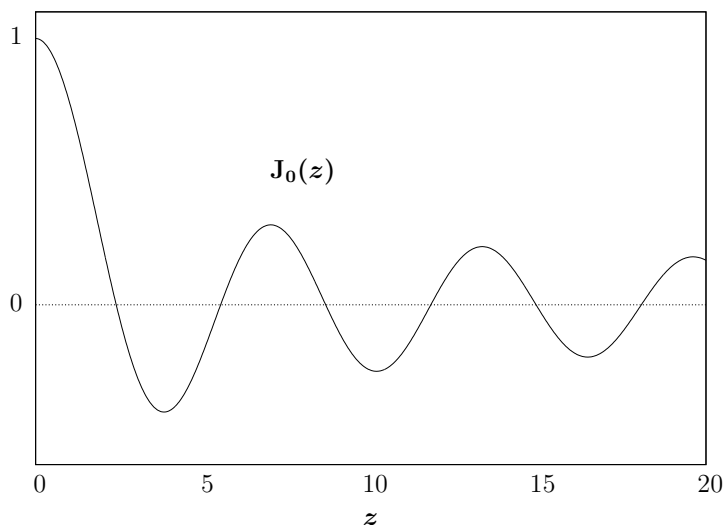
J_0, Y_0 are the first- and second-kind Bessel functions of order zero. In many ways this is very analogous to the oscillator equation:

$$\frac{d^2 y}{dz^2} + y = 0 \quad \rightarrow \quad y(z) = c_1 \cos z + c_2 \sin z$$

J_0, Y_0 are oscillatory functions like the trigonometric functions, and starting with the general solution and applying the given boundary conditions allows us to simplify down to the eigenfunction for specific problems.

BUT there are some important differences too:

1. Bessel's equation has $p(z) = z$, so it is singular at $z = 0$. The $Y_0(z)$ solution is singular there, $Y_0 \rightarrow -\infty$ as $z \rightarrow 0$.
2. Because the oscillator equation is autonomous (its coefficients are not functions of z), its solutions can be written in "shifted" forms, $c_1 \cos z + c_2 \sin z = A \cos(z - z_*)$ (using trig identities). This is not true for Bessel's equation, its solutions CAN NOT be shifted: $J_0(z - z_*)$ is NOT a solution of (1) and there is no identity formula to combine J's and Y's together.
3. J_0, Y_0 are damped oscillator functions and their zeros and critical points (zeros of the first derivative) are not equally-spaced.



Important properties of J_0, Y_0 : For solving BVP's using eigenfunction expansions, we are mainly only interested in the positions of the zeros and critical points of these functions. These values do not have nice formulas, but they have been computerized and tabulated:

For Dirichlet BC's:

<u>zeros of J_0: $J_0(z_k) = 0$</u>		<u>zeros of Y_0: $Y_0(\hat{z}_k) = 0$</u>																												
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For Neumann BC's:

<u>critical points of J_0: $J'_0(\bar{z}_k) = 0$</u>		<table style="border: 1px solid black; padding: 5px;"> <thead> <tr> <th style="text-align: left; padding: 2px;">\bar{z}_k</th> <th style="text-align: left; padding: 2px;">value</th> </tr> </thead> <tbody> <tr><td style="padding: 2px;">\bar{z}_0</td><td style="padding: 2px;">0.000000</td></tr> <tr><td style="padding: 2px;">\bar{z}_1</td><td style="padding: 2px;">3.831706</td></tr> <tr><td style="padding: 2px;">\bar{z}_2</td><td style="padding: 2px;">7.015586</td></tr> <tr><td style="padding: 2px;">\bar{z}_3</td><td style="padding: 2px;">10.173468</td></tr> <tr><td style="padding: 2px;">\bar{z}_4</td><td style="padding: 2px;">13.323692</td></tr> <tr><td style="padding: 2px;"></td><td style="padding: 2px;">...</td></tr> </tbody> </table>	\bar{z}_k	value	\bar{z}_0	0.000000	\bar{z}_1	3.831706	\bar{z}_2	7.015586	\bar{z}_3	10.173468	\bar{z}_4	13.323692		...
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2 Bessel's equation of order m

A generalization of (1) that comes out of separation of variables with coupling to $\{\cos(m\theta), \sin(m\theta)\}$:

$$\boxed{\frac{d}{dz} \left(z \frac{dy}{dz} \right) + \left(z - \frac{m^2}{z} \right) y = 0} \tag{3}$$

Similarly, the general solution is written as

$$\boxed{y(z) = c_1 J_m(z) + c_2 Y_m(z)} \tag{4}$$

J_m, Y_m are similar in spirit to J_0, Y_0 for $m = 0$, but some of their details differ. The zeros and critical points of these functions depend on the value of the m -parameter: there are tables of values for $z_{k,m}, \hat{z}_{k,m}, \bar{z}_{k,m}, \dots$ for each value of $m \geq 0$, $m =$ integers or real numbers too.